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MATHEMATICS OF ACCOUNTING

BY

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PREFACE

This book is designed as a treatise on the subject of Mathematics of Accounting. Much of it has been through two mimeographed editions and thoroughly tested out, and the authors now present it to the public.

In the preparation of this book, two things have been kept constantly in mind—its use as a practical textbook for the student of accountancy, and as a handbook of general information for the accountant—private and public. No pains have been spared to make the solutions, analyses, and illustrations clear and complete in every detail—they show *how* and tell *why*.

The plan of this work is to present, first, the simpler subjects of the mathematics of accounting, and second, the more involved subjects commonly known as the mathematics of investment.

Experience in the classroom teaches that a textbook should furnish ample material for drill on principles, as well as for the application of these principles to practical problems.

The book is presented with the belief that it contains nothing that is not useful to either student or accountant.

Special acknowledgment is due to Mr. Earl G. Lee, public accountant, St. Paul, Minn., who is the author of the chapter on Logarithmic Charts.

STEPHEN F. AUSTIN STATE TEACHERS COLLEGE
NACOGDOCHES, TEXAS

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Part I

Chapter 1

SUGGESTIONS FOR THE ACCOUNTANT

Addition.—Many times, in adding long columns of figures, and carrying from one column to the next, one may be interrupted after having added possibly all except the last column. When the work is resumed it is necessary to re-add the columns previously added in order to pick up the carrying figure. The following plan will save the additional labor of re-adding the columns.

```
 4572
 3986
 2173
 5911
 2765
 4937
------
   24
  32
 40
20
------
24344
```

Explanation.—Add each column separately, setting the sums one place to the left each time as in the illustration. After the last column has been added, add the individual sums in regular order, that is, from right to left. Thus if one is interrupted after adding the third column, it is necessary to add only the fourth column.

Subtraction.—Many errors in subtraction are made when it is necessary to borrow from the next higher order. When that order is reached, one forgets that it has been reduced. These errors can be eliminated to a large extent by changing subtraction to the process of addition, that is, by adding to the minuend the figure required to make it equal the subtrahend.

```
8276
3187
----
5089
```

Explanation.—Instead of saying 7 from 16 is 9, say $7 + 9 = 16$. Write the 9. Add 1, the digit carried over, to the 8, making 9. $9 + 8 = 17$. Write 8, and add 1, the digit carried over, to 1, making 2. $2 + 0 = 2$. Write 0. $3 + 5 = 8$. Write 5. Answer: 5089.

Balancing an account.—Inspection will tell, in most cases, which side is the larger. Add the larger side, put the same footing on the smaller side, leaving space for the balance, then add from the top downward, supplying the figures necessary to make the column total equal the footing previously placed there. The following example will illustrate the method:

Debits		*Credits*
$ 1,956.18		$ 134.26
3,452.75		258.19
289.34		764.83
5,726.31		2,375.94
	Bal.	7,891.36
$11,424.58		$11,424.58

The balance, $7,891.36, was found as follows: Inspection showed the debit side to be the larger in amount. It was added, producing the footing of the account, $11,424.58, which was placed under each column. The right-hand column of the credits adds to 22. Insert 6 to make 28. With 2, the digit carried over, the second column adds to 22. Insert 3 to make 25. The third column and the digit carried over add to 23. Insert 1 to make 24. The fourth column and the digit carried over add to 23. Insert 9 to make 32. The fifth column and the digit carried over add to 16. Insert 8 to make 24. The sixth column and the digit carried over add to 4. Insert 7 to make 11.

Complement method or combined addition and subtraction.—The complement of a number is the amount which, if added to the number, will make one unit of the next higher order. Thus, the complement of 7 is 3, since added they make 10; the complement of 42 is 58, since added they make 100; and the complement of 759 is 241, since added they make 1000.

To find the sum of a column containing numbers to be subtracted, add the complements of the negative or subtrac-

tive items and from the sum of each column deduct as many tens as there are negative numbers in the column.

For an application of this "short cut," take a comparative sales record.

COMPARATIVE SALES RECORD

Dept. No.	*October, 1921*	*October, 1922*	*Increase or Decrease **
A	$ 4,765.18	$ 4,975.83	$210.65
B	3,495.62	3,276.14	219.48 *
C	5,891.18	5,987.53	96.35
D	2,176.95	2,074.16	102.79 *
E	3,984.16	4,176.28	192.12
	$20,313.09	$20,489.94	$176.85

The net increase of all five departments, $176.85, is found by adding the increases and decreases in the following manner: Instead of adding the three numbers showing increases and deducting the sum of the two numbers showing decreases, add the complements of the numbers representing decreases and subtract 20 from the result obtained in adding each column. Instead of $219.48, use the complement of each digit. As $102.79 is a subtractive item, use the complement of each digit.

Begin at the upper right-hand corner and add downward, 5 plus 2 (complement of 8) plus 5 plus 1 (complement of 9) plus 2 equals 15; borrow 10 from the next column to make 25. Subtract 20 and set down the remainder, 5.

Add the second column, remembering that 10 has been borrowed: 6 minus 1, or 5, plus 6 (complement of 4) plus 3 plus 3 (complement) plus 1 equals 18; borrow 10 from the next column, making 28. Subtract 20 and set down 8.

Third column: 1 has been borrowed; therefore, 9 plus 1 (complement) plus 6 plus 8 (complement) plus 2 equals 26. Subtract 20 and set down 6.

Fourth column: 0 (the 1 was borrowed) plus 9 (complement) plus 9 plus 10 (complement of 0) plus 9 equals 37. Subtract 20 and set down the 7. This leaves 1 to carry to the next order.

Fifth column: 1 (carried) plus 2 plus 8 (complement) plus 9 (complement) plus 1 equals 21. Subtract 20 and set down the 1. Answer, $176.85.

Multiplication.—Calculating machines have not eliminated and will not entirely eliminate the necessity of making numerous calculations in the usual manner, that is, mentally or by hand. Anyone having a number of multiplications to make will find it helpful to know the multiplication table up to and including 20×20 instead of up to and including 12×12, as is ordinarily required. Multiplication may be shortened by methods applicable to certain classes of multipliers, etc. A few of the most practical methods are presented in the following paragraphs.

Multiplying by 25.—Annex two zeros to the multiplicand and divide by 4.

Example.—Multiply 7562 by 25.

Solution.

$$4)\ 756200$$
$$189050$$

Multiplying by 15.—Annex a zero to the multiplicand and increase the result by one-half of itself.

Example.—Multiply 8435 by 15.

Solution.

$$84350$$
$$42175$$
$$126525$$

Multiplying numbers ending with zeros.—Multiply the significant figures in each number and to the product annex as many zeros as there are zeros in both the multiplier and the multiplicand combined.

Example.—Multiply 756,000 by 4200.

Solution.

$$756$$
$$42$$
$$31752$$

Annex 5 zeros Answer: 3,175,200,000

Multiplication by numbers near 100, as 98, 97, 96, etc., and numbers near 1000, as 997, 996, etc.—This method is of

value in finding the net proceeds of some amount less 2%, 3%, etc., in fact there are many times when the above numbers are used as multipliers.

Example.—Multiply 3247 by 97.

Solution.

Multiply the number by 100 and subtract 3 times the number.

$$\begin{aligned} 324700 &= 3247 \times 100 \\ 9741 &= 3247 \times 3 \\ \hline 314959 &= 3247 \times 97 \end{aligned}$$

Multiplication by a number near 1000 is carried out in the same manner by multiplying by 1000, instead of by 100.

Multiplication of two numbers each near 100, 1000, etc.

Example.—Multiply 96 by 98.

Solution.

	(comp)
96	4
98	2
9408	

Explanation.—Step 1. Multiply the complements of the two numbers, and if the product occupies units' place only, prefix a zero.

Step 2. Subtract the complement of one number from the other number and write the result at the left of the result in Step 1. The complement of either number subtracted from the other number leaves the same remainder, as 96 — 2 or 98 — 4 each gives 94. Answer: 9408.

Example.—Multiply 92 by 88.

Solution.

	(comp)
88	12
92	8
8096	

Explanation.—The product of the complements is 96, the last two figures of the answer. 88 — 8 or 92 — 12 = 80, the first two figures of the answer. Answer: 8096.

Preparation of tables of multiples of multipliers.—It is not uncommon to have to use the same multiplier many times in making calculations, especially in cost accounting. A saving in time, and increased accuracy of the work are the results if a table of multiples of the multiplier is constructed. Suppose there are a number of multiplications in which 326,834 is to be used as the multiplier. This table may be constructed with an adding machine by locking the repeat key. Sub-total after each pull of the handle. The sub-totals should check with the product column. If the table is prepared by repeated additions, the 10th product should be computed as it will verify all, unless there are compensating errors in the work.

Multiplier		*Product*
1	.....	326834
2	(326834 + 326834).....	653668
3	(653668 + 326834).....	980502
4	(980502 + 326834).....	1307336
5	(1307336 + 326834).....	1634170
6	(1634170 + 326834).....	1961004
7	(1961004 + 326834).....	2287838
8	(2287838 + 326834).....	2614672
9	(2614672 + 326834).....	2941506
Verification:		
10	(2941506 + 326834).....	3268340

Example.—Multiply 326,834 by 5249.

Solution.

2941506 = 9 times 326834
1307336 = 4 " "
653668 = 2 " "
1634170 = 5 " "
1715551666 = Product

Use of tables in division.—If a number of divisions is to be made with the same divisor, it is advantageous to set up a table of multiples of the divisor.

Example.—Assume that 328 is to be used a number of times as a divisor, and that one of the dividends is 587954.

SET-UP OF MULTIPLES

Multiplier	*Product*
1	328
2	656
3	984
4	1312
5	1640
6	1968
7	2296
8	2624
9	2952

Solution.

328) 587954 (1791 89/164
328
2599
2296
3035
2952
834
656
178

$\frac{178}{328} = \frac{89}{164}$

Explanation.—Inspection shows the first digit in the quotient to be 1. The second partial dividend is 2599. The product column shows the largest product contained therein to be 2296, opposite 7. The third partial dividend is 3035, and the product column shows the largest product contained therein to be 2952, opposite 9. The fourth partial dividend is 834 and the largest product contained therein is 656, opposite 2. The remainder is 178. This remainder may be reduced to a common fraction as shown, or it may be computed as a decimal.

Division in this manner is rapid, as no time is lost through selection of a quotient so large that when the product is found it exceeds the dividend, necessitating another trial.

Reciprocals in division.—The reciprocal of any number is found by dividing 1 by the number. The reciprocal of 5 is $1 \div 5$ or .2, and the reciprocal of 25 is $1 \div 25$ or .04.

The quotient in a division can be found by multiplying the dividend by the reciprocal of the divisor. Those who use calculating machines will find this method valuable.

Example.—Divide 47568 by 125.

Solution.

The reciprocal of 125 is $1 \div 125$ or .008. $47568 \times .008 = 380.644$.

Use of reciprocals in cost accounting.—In calculating costs by the piece, job, or department where much percentage work is involved, and in the preparation of pay-rolls, burden statements, and expense statements, reciprocals will save many tedious divisions.

The following example illustrates one of the several uses of reciprocals in cost work:

Example.—Find the per cent that each department's monthly expense is of the total monthly expense.

Department	*Expense*
A	$ 600
B	500
C	1200
D	700
E	1000
Total	$4000

Solution.

Divide $1 by $4000 to obtain the reciprocal .00025. Multiply the expense of each department by this reciprocal, and the product will be the per cent the department's expense is of the total expense.

Department	*Expense*	*Reciprocal*	*Per Cent*
A	$ 600	.00025	15%
B	500	"	12½%
C	1200	"	30%
D	700	"	17½%
E	1000	"	25%
Total	$4000		100%

To abbreviate decimal multiplication when a given number of decimal places is required.—

Example.—Multiply 4.7892 by 3.1765, and obtain the answer correct to four decimal places.

Solution.

4.7892		multiplicand
5 6713		multiplier reversed.
14.3676		$= 4.7892 \times 3.$
.4789	2	$= 4.789\not{2} \times .1$
.3352	4 $\not{4}$	$= 4.78\not{9}\not{2} \times .07$
. 287	3 $\not{3}\not{2}$	$= 4.7\not{8}\not{9}\not{2} \times .006$
. 23	9 $\not{4}\not{3}\not{9}$	$= 4.\not{7}\not{8}\not{9}\not{2} \times .005$
15.2128	9 $\not{3}\not{4}\not{9}$	

Answer = 15.2128

Explanation.—The multiplier, 3.1765, is written in the reverse order, 56713, the units' digit being placed under the lowest order of the multiplicand that is desired in the product —ten thousandths. Multiply by each digit of the reversed multiplier, beginning with that digit of the multiplicand which stands directly above the digit of the multiplier used, taking care to include the digit carried over obtained by multiplying one (or two) rejected digits at the right.

Example.—Multiply 4.7869347 by 7.25, and obtain the quotient correct to three decimal places.

Solution.

4.786 9347		
527		
33.508	3	$= 4.786\not{9} \times 7$
.957	2	$= 4.78\not{6}\not{9} \times 2$
.239	0	$= 4.7\not{8}\not{6}\not{9} \times 5$
34.704	5	

Answer = 34.704

Division of decimals.—Division of decimals may often be abbreviated to advantage, especially when the divisor is given to a greater number of decimal places than are contained in the dividend.

Example.—Divide 4.39876 by 2.4871934, and obtain the quotient correct to three decimal places.

Solution.

Ordinary Method.

```
2.4871934) 4.398 7600 (1.768
           2 487 1934
           -----------
           1 911 56660
           1 741 03538
           -----------
             170 531220
             149 231604
             ----------
              21 2996160
              19 8975472
              ----------
               1 4020688
```

Answer = 1.768.

Abridged Method.

```
2.487 1934) 4.398 7 (1.768
            2 487 2
            -------
            1 911 5
            1 741 0
            -------
              170 5
              149 2
              -----
               21 3
               19 9
               ----
                1 4
```

Answer = 1.768.

Explanation.—By observing the ordinary method, it is apparent that the third decimal place in the quotient is not affected by the digit in the third decimal place in the divisor (except by the carrying digits).

Since the units' digit of the divisor is contained in the units' digit of the dividend, the first digit in the quotient is in the units' place, and as three decimal places are required, the quotient will contain four digits. Therefore, the last four digits of the divisor will not affect the quotient, except through the digits carried over.

The first four digits of the divisor, 2.487, are contained once in 4.398. Multiplying that part of the divisor used, by the quotient digit (including the digit carried over from one or two following digits—in this case considering the 9 as a unit and adding it to the 1 making 2) gives 2487 2 and this result deducted from the previous dividend leaves 1911 5 for a new dividend.

Cancel the right-hand digit, 7, of the divisor, and divide 1911 by 248 which gives the quotient 7. Multiplying the divisor by 7 (including carrying digit) gives 1741 0, and subtracting leaves a new dividend of 170 5.

Cancel another digit, 8, of the divisor and divide by 24. It is contained 6 times in 170. The product (including the digit carried over) is 149 2, and this product subtracted leaves a new dividend of 21 3.

Cancel another digit, 4, of the divisor. Divide 21 by 2, using the carried digit; the result is 8. The new product is 19 9, and this product subtracted from 21 3 leaves a remainder of 1 4.

Example.—Divide 8.47 by 31.76983476, and obtain the quotient correct to three decimal places.

31.769 83476) 8.4700	(.2665		
6.3540 (6)	=	31.769	(8) × .2 with (6) turned into 1 more.
2 1160			
1 9062 (9)	=	31.76	(9) × .06 with (9) turned into 1 more.
2098			
1906 (1)	=	31.7	(6) × .006
182			
159 (8)	=	31.	(7) × .0005 with (8) turned into 1 more.
23			

Answer = *.2665.*

Problems.

1. Multiply by 25:

3456
6782
6842
9813
8456

2. Multiply by 15:

5134
1243
6578
5618
6157

3. Multiply:

345600 by 4200 _____
8600 by 4700 _____
450 by 280 _____
12340 by 2300 _____
4560 by 1800 _____

614200 by 20 _____
7500 by 3100 _____
170 by 2500 _____
931000 by 450 _____
761200 by 280 _____

4. Multiply:

3456 by 97 _____
2345 by 98 _____
6759 by 92 _____
4567 by 91 _____
6712 by 94 _____

6713 by 95 _____
8412 by 96 _____
6792 by 93 _____
9456 by 99 _____
4272 by 97 _____

5. Multiply:

4532 by 998	_____	4123 by 994	_____
4513 by 992	_____	9231 by 995	_____
5693 by 991	_____	4153 by 996	_____
5692 by 997	_____	5143 by 999	_____
6145 by 993	_____	3756 by 995	_____

6. Multiply:

98 by 93	_____	93 by 94	_____
92 by 91	_____	99 by 99	_____
95 by 95	_____	98 by 98	_____
94 by 92	_____	97 by 97	_____
96 by 97	_____	96 by 96	_____

7. Multiply:

998 by 995	_____	997 by 997	_____
992 by 994	_____	996 by 992	_____
995 by 991	_____	998 by 991	_____
993 by 993	_____	993 by 997	_____
995 by 994	_____	992 by 998	_____

8.

COMPARATIVE SALES RECORD

Dept. No.	*November, 1922*	*November, 1923*	*Increase or Decrease **
1	6,834.78	7,134.95	____
2	5,764.87	6,125.19	____
3	8,275.16	7,934.97	____
4	5,986.35	6,344.76	____
5	3,756.89	3,695.14	____
6	9,375.18	9,567.98	____
7	8,467.56	8,567.39	____
8	4,865.84	5,607.18	____
9	10,785.65	11,265.39	____
10	12,764.16	14,572.86	____
Total			____

9. In the table below, add the Goods on Hand, June 1, 1922, to the Purchases for the Year, and from this sum subtract the Goods on Hand, June 1, 1923. This gives the Cost of Goods Sold. The operation should be performed without transferring any of the figures. Use the complement of the numbers in the column "Goods on Hand, June 1, 1923."

The difference between the Cost of Goods Sold and the Sales will give the Profit or Loss.

To verify the work, add all of the columns and deal with the totals in the same way as with the figures for the departments. The difference between the total Cost of Goods Sold column and the Sales column should equal the difference be-

tween the Profit and Loss columns, showing the Net Profit of the ten departments for the year.

GROSS PROFIT BY DEPARTMENTS

Dept.	*Goods On Hand June 1, 1922*	*Purchases for the Year*	*Goods On Hand June 1, 1923*	*Cost of Goods Sold*	*Sales*	*Profit*	*Loss*
1	2,475.86	9,846.37	2,347.11	—	12,678.92	—	—
2	1,357.10	6,275.40	1,475.86	—	6,288.90	—	—
3	3,276.84	10,326.85	3,127.84	—	8,297.63	—	—
4	5,475.90	11,176.98	4,874.13	—	13,586.47	—	—
5	4,176.83	9,798.34	4,207.16	—	10,508.92	—	—
6	3,785.47	8,276.41	3,684.10	—	8,756.13	—	—
7	2,986.17	9,386.57	3,014.72	—	8,964.85	—	—
8	3,275.83	8,724.18	2,817.56	—	9,675.34	—	—
9	2,976.94	9,543.34	2,734.15	—	10,789.38	—	—
10	3,532.25	10,217.60	3,275.89	—	12,756.84	—	—
Footings							

10. Multiply:

(a) 6.789452 by 7.176 obtaining the product correct to the 4th decimal.
(b) 8.596 2/3 by 24.75 " " " " " " 3rd "
(c) 176.85 2/7 by 2.764 " " " " " " 4th "
(d) 29.715643 by 85.16 " " " " " " 3rd "
(e) 59.387426 by 8.1475 " " " " " " 3rd "

11. Divide:

(a) 3.769 by 25.786342 obtaining the quotient correct to the 3rd decimal.
(b) 85.143 by 5.4976348 " " " " " " 4th "
(c) 1.765489 by 24.139876 " " " " " " 3rd "
(d) 80.70854 by 9.3497658 " " " " " " 4th "
(e) 2.769834 by .487693287 " " " " " " 3rd "

12. Find the per cent of floor space each department occupies:

Department	*Floor Space*	*Per Cent*
Grinding	2250 sq. ft.	——
Buffing and Plating.....	3375 " "	——
Assembling	7000 " "	——
Packing	6250 " "	——
Stores	3500 " "	——
Office	2625 " "	——
Total	25000 " "	

13. Find the reciprocal of 144 and change the following units to gross lots.

4032 36288 570528

14. Set up a table of multiples of 144. Using the table, find by division the gross lots in the preceding problem.

Chapter 2

VERIFICATION OF COMPUTATIONS BY CHECK NUMBERS

Check numbers obtained by casting out the nines.—A common method of verifying arithmetical computations is by casting out the nines (finding the excess of nines in a number). This method is the simplest of all methods of verification and may be used to good advantage in many cases.

Verification of addition.—

Example.

8342	8
8967	3
8378	8
9276	6
8431	7
43394–5	32–5

Solution.

Upon adding the digits of 8342, we obtain 17. Cast out 9 and set down 8. If a number contains a 9, skip it in adding the digits; thus in 8967, $8 + 6 + 7$ equals 21. Cast out the nines and set down the excess, 3. Find the check number of each number in the same way. Add the check numbers and cast the nines out of their sum. Find the check number of the sum of the column being verified. The final check number is 5 in each case.

Verification of subtraction.—

Example.

7856	8
2138	5
5718	3

Solution.

7856 checks 8, and 2138 checks 5. $8 - 5 = 3$, and 5718 checks 3.

Verification of multiplication.—

Example.

482	5
376	7
181232–8	35–8

Solution.

482 checks 5, and 376 checks 7. $7 \times 5 = 35$. 35 checks 8, and the product, 181,232, also checks 8.

Verification of division.—Division may be verified by multiplication; that is, the product of the quotient and the divisor should equal the dividend. Apply the same principle in verifying with check numbers.

Example.

```
13) 76492 (5884
    65
    ---
    114
    104
    ---
     109
     104
     ---
      52
      52
      --
```

Solution.

76492 checks 1. 13 checks 4. 5884 checks 7. $4 \times 7 = 28$, and 28 checks 1, which is also the check number of the dividend.

Verification of division where there is a remainder.—The check number of the remainder, plus the product of the check number of the quotient, multiplied by the check number of the divisor, should equal the check number of the dividend.

Example.

```
32) 75892 (2371
    64
    ---
    118
     96
     ---
     229
     224
     ---
       52
       32
       --
       20
```

Solution.

Step 1. The remainder, 20, checks 2.
The quotient, 237, checks 4.
The divisor, 32, checks 5.
$2 + (4 \times 5) = 22$.
22 checks 4.

Step 2. The dividend, 75892, checks 4.

Step 1 and Step 2 should produce the same check number.

Check numbers obtained by casting out the elevens.— A great many people use eleven instead of nine, as a check number, since the latter will not detect two transposed digits.

Begin with the left-hand digit of the first number and subtract it from the digit at its immediate right. If the digit to the right is smaller, add eleven, before subtracting. Using the remainder as a new digit, subtract it from the third digit from the left of the number, first adding eleven if it is necessary. Use this remainder as a new digit and subtract it from the fourth digit from the left, first adding eleven if necessary. Continue in this manner until all the digits in the number have been used. The final remainder is the check number of the number.

Another method of check-numbering by means of the figure eleven uses alternate digits. The sum of the first, third, and fifth digits (beginning at the right) is subtracted from the sum of the second, fourth, and sixth digits. If the subtraction cannot be performed, eleven is first added to the sum of the even digits, and the sum of the odd digits is deducted, the remainder being the check number.

Verification of addition.—

Example.

4324	1
8689	10
6327	2
8964	10
3487	0
31791–1	23–1

Solution.

Begin at the left with the number, 4324. 4 from 14 $(3 + 11) = 10$. 10 from 13 $(2 + 11) = 3$. 3 from $4 = 1$.

Take the second number, 8689. 8 from 17 $(6 + 11) = 9$. 9 from 19 $(8 + 11) = 10$. 10 from 20 $(9 + 11) = 10$. Check all the numbers and in the same manner. Add the check numbers. The sum of the check numbers checks 1, and the sum of the numbers checks 1.

Verification of subtraction.—

Example.

7453	6
1289	2
6164	4

Solution.

7453 checks 6. 1289 checks 2. $6 - 2 = 4$ and 6164 checks 4.

Verification of multiplication.—

Example.

584	1
256	3
149504	3

Solution.

584 checks 1. 256 checks 3. $3 \times 1 = 3$ and 149,504 checks 3.

Verification of division.—

Example (a).

```
24) 89784 (3741
    72
    ---
    177
    168
    ---
     98
     96
     --
     24
     24
     --
```

Solution.

89,784 checks 2. 24 checks 2. 3741 checks 1. $2 \times 1 = 2$, the check figure of the dividend.

Example (b).

```
31) 75893 (2448
    62
    ---
    138
    124
    ---
     149
     124
     ---
      253
      248
      ---
        5
```

Solution.

75,893 checks 4. 31 checks 9. 2448 checks 6. The remainder checks 5. $5 + (9 \times 6) = 59$. 59 checks 4, the same check number as that of the dividend.

Problems.

1. Add and verify by casting out the nines:

3478	7957	7564	5146
3192	8245	3472	7451
8402	9124	6382	1248
6713	5412	9245	9365
1362	8417	7645	8216
8124	9219	8216	8419
6457	3568	9315	9245
8214	7213	7215	7432
1869	1243	7453	9126
9121	6534	2419	6532

2. Add and verify by casting out the elevens:

45674–2	82134
34156–1	95613
92345–0	81452
18679	14967
14537	81234

3. Multiply and verify by casting out the nines:

385	754	275	475
263	189	321	286

4. Multiply and verify by casting out the elevens:

498	758	891	456
213	284	278	584

5. Subtract and verify by casting out the nines:

3472	7850	7694
1456	2453	1389

6. Subtract and verify by casting out the elevens:

4857	9758	7684
1324	4259	3075

7. Divide and verify by casting out the nines:

46) 789682 (____ 51) 29765 (____

39) 897663 (____ 47) 593764 (____

8. Divide and verify by casting out the elevens:

32) 17696 (____ 86) 53979 (____

57) 81738 (____ 126) 4786432 (____

Chapter 3

FRACTIONS

Fundamentals.—Ordinary addition, subtraction, multiplication and division of fractions should be understood by the reader of this chapter before taking up this text. Anyone not familiar with these fundamental processes should review fractions in a commercial arithmetic.

In this chapter attention is directed to certain labor-saving devices in the use of common fractions and mixed numbers.

To square a mixed number whose fractional part is ½.—

Example.—Multiply 4½ by 4½.

Solution.

$4\frac{1}{2} \times 4\frac{1}{2} = 20\frac{1}{4}$.

Explanation.—$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, the fractional part of the answer. Since ½ of 4 is to be taken two times, it is equivalent to 1×4; therefore, increase the multiplier by one unit and multiply. $5 \times 4 = 20$. Answer: 20¼.

To square a mixed number whose fractional part is ¼.—

Example.—Multiply 6¼ by 6¼.

Solution.

$6\frac{1}{4} \times 6\frac{1}{4} = 39\frac{1}{16}$.

Explanation.—$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$, the fractional part of the answer. $6 \times 6 = 36$, and to this product add ¼ (6 + 6) or 3, making a total of 39. Answer: $39\frac{1}{16}$.

To multiply a mixed number by a mixed number.—

Example.—Multiply 524½ by 27⅓.

Solution.

524½		
27⅓		
14148	6	= Common denominator.
174⅔	4	= Numerators of changed fractions.
13½	3	
⅙	1	
14336⅓	⅔	= 1⅓

Explanation.—Multiply 524 by 27, obtaining the first part of the answer, 14148. Next, take ⅓ of 524, obtaining 174⅔. Then take ½ of 27, obtaining 13½. Finally take ⅓ of ½, obtaining ⅙. Add the four partial products and the complete product is 14,336⅙.

The use of aliquot parts.—To know the decimal equivalent of a common fraction, or conversely, to know the equivalent common fraction for a decimal fraction saves a great deal of time in multiplication and in division. Aliquot parts are of value in addition and subtraction if an adding machine or a calculating machine is used, because machines are not adapted for work involving common fractions.

TABLE OF ALIQUOT PARTS OF 1

Common Fraction	*Decimal Equivalent*	*Common Fraction*	*Decimal Equivalent*
½	.50	⅑	.11⅑
⅓	.33⅓	1/10	.10
⅔	.66⅔	1/11	.09 1/11
¼	.25	1/12	.08⅓
¾	.75	5/12	.41⅔
⅕	.20	7/12	.58⅓
⅙	.16⅔	11/12	.91⅔
⅚	.83⅓	1/15	.06⅔
1/7	.14 2/7	1/16	.06¼
2/7	.28 4/7	3/16	.18¾
3/7	.42 6/7	5/16	.31¼
4/7	.57 1/7	7/16	.43¾
5/7	.71 3/7	9/16	.56¼
6/7	.85 5/7	11/16	.68¾
⅛	.12½	15/16	.93¾
⅜	.37½	1/25	.04
⅝	.62½	1/32	.03⅛
⅞	.87½	3/32	.09⅜

The fractions can be carried out as decimals, as the work demands.

Multiplication by aliquot parts.—

Example.—Find 16⅔% of $475.34.

Solution.

6) $475.34

$79.22

Explanation.—Since .16⅔ equals ⅙, divide $475.34 by 6.

Example.—Find the cost of 256 articles at 37½¢ each.

Solution.

256 × $1 = $256

8) $256

32

3

$96

Explanation.—37½¢ is ⅜ of $1. 256 articles at $1 would cost $256. Since 37½¢ is ⅜ of $1, ⅜ of $256 is the cost, or $96.

Division by aliquot parts.—It is difficult to divide a number by a mixed number. If the divisor is an aliquot part, the quotient can be found by multiplication as follows:

Example.—Divide 4875 by 16⅔.

Solution.

48.75

6

292.50

Explanation.—Since 16⅔ is ⅙ of 100, divide 4875 by 100 by pointing off two decimal places from the right, and multiplying the answer by 6. Point off two decimal places in the product and the answer is 292.5 or 292½.

Example.—The production cost of 1250 units is $3170. Find the cost per unit.

Solution.

.3170

8

$2.5360

Explanation.—1250 is ⅛ of 10,000. Divide $3170 by 10,000 by pointing off 4 decimal places from the right, and multiply by 8. The cost per unit is $2.536.

Problems.

1.

3½ × 3½ = _____

55 × 55 = _____

2¼ × 2¼ = _____

8¼ × 8¼ = _____

425 × 425 = _____

2.

456½ × 23¼ = _____

348⅛ × 74⅓ = _____

1748 × 12½ = _____

345 × 33⅓ = _____

1428 × 28⅐ = _____

3. Express as decimal fractions: *

1/3 _____	1/16 _____	5/7 _____	6/7 _____
1/4 _____	1/20 _____	7/8 _____	5/8 _____
1/5 _____	1/25 _____	3/4 _____	3/32 _____
1/6 _____	1/32 _____	5/12 _____	1/11 _____
1/7 _____	2/5 _____	3/16 _____	1/14 _____
1/8 _____	3/5 _____	2/9 _____	4/9 _____
1/9 _____	4/5 _____	3/7 _____	1/24 _____
1/10 _____	5/6 _____	5/16 _____	5/32 _____
1/12 _____	2/7 _____	2/3 _____	7/8 _____
1/15 _____	3/8 _____	8/9 _____	7/12 _____

* Non-terminating ones should be carried to the sixth decimal place and the common fraction annexed.

4. Extend the following problems mentally (by use of aliquot parts):

24 at .12½ _____	48 at .06¼ _____
32 at .25 _____	55 at .09 1/11 _____
16 at .75 _____	24 at .62½ _____
48 at .18¾ _____	36 at .41⅔ _____
45 at .11 1/9 _____	32 at .43¾ _____
56 at .62½ _____	48 at .37½ _____
64 at .03⅛ _____	18 at .44 4/9 _____
96 at .16⅔ _____	75 at .06⅔ _____
75 at .33⅓ _____	84 at .08⅓ _____
64 at .56¼ _____	96 at .16⅔ _____
84 at .58⅓ _____	54 at .83⅓ _____
27 at .22 2/9 _____	75 at .25 _____
48 at .37½ _____	96 at .75 _____
24 at .08⅓ _____	12 at .12½ _____
9 at .66⅔ _____	14 at .42 6/7 _____
30 at .06⅔ _____	28 at .57 1/7 _____
35 at .14 2/7 _____	14 at .07 1/7 _____
49 at .28 4/7 _____	8 at .75 _____
72 at .83⅓ _____	56 at .87½ _____
96 at .87½ _____	32 at .12½ _____

5. Divide:

4876 by 12½.
5134 by 16⅔.
4852 by 11 1/9.
3876 by 125.

Chapter 4

PERCENTAGE

Statistics.—Business operations are guided by means of carefully prepared statistics, and the relationships of items in statistics are more clearly reflected when expressed in terms of percentage.

Definitions.—The *base* is the number of which so many hundredths are taken; for example, total sales, total expense, amount of a note, par value of a bond, and so forth.

The *rate* is the number denoting how many hundredths of the base are taken; for example, 6%, 25%, and so forth, written decimally as .06, .25, and so forth.

The *percentage* is the result obtained by taking so many hundredths of the base; for example, interest cost on a sum of money, departmental portions of expense items, and so forth.

Fundamental processes.—The percentage is the product of the base times the rate.

$$\text{Base} \times \text{Rate} = \text{Percentage}.$$

Example.—6% interest on a \$1000 bond = \$60.

The base may be found by dividing the percentage by the rate.

$$\text{Percentage} \div \text{Rate} = \text{Base}.$$

Example.—\$60 ÷ .06 = \$1000.

The rate may be found by dividing the percentage by the base.

$$\text{Percentage} \div \text{Base} = \text{Rate}.$$

Example.—\$60 ÷ \$1000 = .06 or 6%.

Problems.

1. Find the rates per cent in the statement of sales below. To make the calculations easier, the nearest number of dollars is taken. If a salesman's sales amount to \$1264.85

use $1265. The sales in the problem have been set up to conform to this practice.

The total sales constitute the base, and each salesman's sales is a percentage.

PER CENT OF MONTH'S SALES MADE BY EACH SALESMAN

Salesmen's Numbers	*Sales*	*Per Cent of Total*
50	$1,265.00	——
51	1,914.00	——
52	1,733.00	——
53	1,627.00	——
54	1,692.00	——
55	2,138.00	——
56	1,728.00	——
57	1,595.00	——
Total		100.00%

2. Find the per cent that each product is of the total amount manufactured.

Product	*Amount Mfg.*	*Rate Per Cent*
#1	187,515 pounds	——
#2	246,735 "	——
#3	565,467 "	——
#4	216,738 "	——
Total		100.00%

3. Find the per cent that each department's expense is of the total expense.

	Expense	*Per Cent*
Department A	$ 125.00	——
" B	250.45	——
" C	1,255.55	——
" D	650.35	——
" E	740.25	——
Total		100.00%

4. Find the per cent of Total Gross for each department.

NORTHERN STATES POWER COMPANY AND SUBSIDIARIES
COMPARATIVE GROSS EARNINGS BY DEPARTMENTS

	Electric Department		Gas Department		Steam Department	
Year	*Gross*	*Per cent of Total Gross*	*Gross*	*Per cent of Total Gross*	*Gross*	*Per cent of Total Gross*
1921	11,164,308	——	705,959	——	648,321	——
1920	10,532,234	——	660,594	——	687,371	——
1919	8,797,249	——	523,120	——	628,816	——
1918	7,527,335	——	436,620	——	512,151	——

(*Continued* on page 28.)

NORTHERN STATES POWER COMPANY AND SUBSIDIARIES
COMPARATIVE GROSS EARNINGS BY DEPARTMENTS—*Cont.*

	Street Railway Department		Telephone Department		All Departments	
Year	*Gross*	*Per cent of Total Gross*	*Gross*	*Per cent of Total Gross*	*Gross*	*Per cent*
1921.....	362,380	——	82,254	——	12,963,222	100
1920.....	356,324	——	75,723	——	12,312,246	100
1919.....	315,001	——	68,015	——	10,332,201	100
1918.....	252,671	——	66,049	——	8,794,826	100

5. In the budget given below, find the per cent that each budget expenditure is of the total budget.

MUNICIPAL BUDGET

WATERWORKS DEPARTMENT

Water Department	*Amount*	*Per Cent of Total Budget*
Pump Station and Filter Plant Salaries.....	$17,300.00	——
Office Salaries and Expenses.....	4,600.00	——
Chemicals, Filter Plant.....	1,000.00	——
Power—Pump Station and Filter Plant.....	15,000.00	——
Light, Heat and Supplies.....	3,000.00	——
Water Service	3,000.00	——
Meters and Installation.....	6,000.00	——
Water Main Extensions and Fire Hydrants....	3,000.00	——
Motor Truck Repairs.....	150.00	——
Interest on Outstanding Warrants.....	4,246.00	——
Total	$57,296.00	——
Public Buildings and Utilities		
City Hall Engineers and Janitors.....	$ 5,060.00	——
City Hall Fuel and Supplies.....	4,000.00	——
City Hall Maintenance and Repairs.....	1,000.00	——
Detention Hospital Repairs.....	300.00	——
Detention Hospital Lights and Fuel.....	900.00	——
Park Light and Fuel.....	1,200.00	——
Septic Tank Electric Power.....	600.00	——
Septic Tank Repairs.....	100.00	——
Incinerator Fuel and Light.....	1,000.00	——
Electric Lighting—Streets, Alleys, and White Way	14,500.00	——
Interest on Warrants.....	2,284.00	——
Contingent Fund	5,000.00	——
Total	$35,944.00	——
Summary		
Water Department	$57,296.00	——
Public Buildings and Utilities.....	35,944.00	——
Total Budget	$93,240.00	——

6. In the report given below, find:
   (*a*) Total expenses for each day.
   (*b*) Total all columns.
   (*c*) Per cents that expenses are of sales for each day and per cent of total expense to total sales.

SALESMEN'S WEEKLY EXPENSE REPORT WEEK ENDING NOVEMBER 17, 1923

Day	*Break-fast*	*Lunch*	*Dinner*	*Total Meals*	*Bed*	*Total*	*Rail-road*	*Laun-dry*	*Bus*	*Miscel.*	*Total*	*Sales*	*Per Cent*
Sunday	.35	.60	.75	1.70	2.00	3.70					3.70		
Monday	.35	.50	.50	1.35	2.00	3.35	1.75		.25	.65	6.00	$120	5%
Tuesday	.35	.50	.60	——	2.00	——	1.90		.25	.75	——	175	
Wednesday	.35	.50	1.50	——	2.50	——	1.50		.25	1.15	——	200	
Thursday	.25	.60	.75	——	2.25	——	1.65		.50	.60	——	150	
Friday	.35	.65	.60	——	2.00	——	2.10		.25	.20	——	180	
Saturday	.40	.55	.60	——	2.00	——	1.85		.25	.30	——	130	
Total													

Cash on hand at beginning of week $56.75

Expenses for week _____

Balance on hand (or due) at end of week _____

7. Find (*a*) the total sales for each day in all departments (add across); (*b*) the total sales for each department for the week (add downward); (*c*) in two ways the total sales in all departments for the entire week; (*d*) the per cent of grand total sales made each day (total for each day divided by the grand total); (*e*) the per cent of grand total sales made in each department (total of each department divided by the grand total).

WEEKLY SALES IN A DEPARTMENT STORE

Day	*Dept. A*	*Dept. B*	*Dept. C*	*Total*	*Per Cent*
Monday	475.86	324.18	456.19		
Tuesday	275.83	174.82	259.80		
Wednesday	329.86	274.19	179.86		
Thursday	424.83	285.27	258.24		
Friday	387.92	304.14	286.39		
Saturday	412.15	319.28	305.74		
Total.....					
Per Cent	%	%	%		100.00%

8. Find the per cent of increase in the figures below.

INDUSTRIAL STATISTICS OF THE SOUTHERN STATES TRAVERSED BY SOUTHERN RAILROAD

	1920	*1905*	*% Increase*
Coal Production (tons)—			
Southern States traversed by Southern Railway	57,410,715	27,405,447	
Other States of the country.....	519,020,535	323,239,763	
Pig Iron Production (tons)—			
Southern States traversed by Southern Railway	3,877,850	2,589,398	
Other States of the country.....	33,048,137	20,402,982	
Cotton Consumption (bales)—			
Southern States traversed by Southern Railway	3,504,667	2,062,230	
Other States of the country.....	3,257,540	2,216,750	
Cotton Spindles (No.)—			
Southern States traversed by Southern Railway	14,932,919	7,920,715	
Other States of the country.....	20,901,544	16,156,809	

9. Find (*a*) the increase or decrease in monthly sales by departments; (*b*) each department's per cent of increase or decrease (divide increase or decrease in each department by that department's monthly sales for 1920).

COMPARATIVE SALES SHEET

PER CENT OF INCREASE OR DECREASE

Dept.	*Total Sales, Jan. 1920*	*Total Sales, Jan. 1921*	*Increase or Decrease **	*% Increase or Decrease **
A	$2,896.34	$2,974.59	——	——
B	1,752.18	1,526.34	——	——
C	3,475.64	3,679.82	——	——
D	2,578.15	2,698.51	——	——
E	1,976.39	2,078.95	——	——
Total				

* Indicate decreases with asterisks.

10. Find the per cents of increase in the following statistics:

SOUTHERN RAILWAY COMPANY

TRAFFIC STATISTICS OF 1920 COMPARED WITH 1905

	1920	*1905*
Average miles operated.....	6,973.17	7,198.99
Gross earnings	$152,817,410	$48,145,108
Gross earnings per mile.....	$21,915	$6,687
Revenue tons carried one mile.....	8,236,072,484	3,365,362,765
Revenue tons carried one mile per mile of road	1,181,109	467,477
Average tons per revenue freight train-mile	461	194
Average earnings per revenue freight train-mile	$5.62	$1.83
Passengers carried one mile.....	1,229,054,088	545,795,683
Passengers carried one mile per mile of road	176,255	75.815
Average passengers per train-mile.....	69	35
Average earnings per passenger train-mile	$2.62	$1.01

11. In the following budget, find the amount of increase or decrease in the 1924 budget over the 1923 budget and the per cent of increase or decrease.

COMPARATIVE ANNUAL BUDGET OF A MUNICIPALITY
WATERWORKS DEPARTMENT—YEARS 1923 AND 1924

WATER DEPARTMENT	*1924*	*1923*	*Increase or Decrease**	*Per Cent of Increase or Decrease**
Pump Station and Filter Plant Salaries	$17,300.00	$14,716.00	——	——
Office Salaries and Expense	4,600.00	3,850.00	——	——
Chemicals, Filter Plant	1,000.00	1,000.00	——	——
Power, Pump Station and Filter Plant	15,000.00	15,000.00	——	——
Light, Heat, and Supplies	3,000.00	3,000.00	——	——
Water Service	3,000.00	2,700.00	——	——
Meters and Installation	6,000.00	2,000.00	——	——
Water Main Extensions and Fire Hydrants	3,000.00	2,500.00	——	——
Motor Truck Repairs	150.00	25.00	——	——
Interest on Outstanding Warrants	4,246.00	3,750.00	——	——
New Pump Equipment and Installation		4,000.00	——	——
	$57,296.00	$52,541.00	——	——
PUBLIC BUILDINGS AND UTILITIES				
City Hall Engineers and Janitors	$ 5,060.00	$ 4,284.00	——	——
City Hall Fuel and Supplies	4,000.00	2,216.00	——	——
City Hall Maintenance and Repairs	1,000.00	850.00	——	——
Detention Hospital Repairs	300.00	400.00	——	——
Detention Hospital Light and Fuel	900.00	700.00	——	——
Park Light and Fuel	1,200.00	1,050.00	——	——
Septic Tank Electric Power	600.00	600.00	——	——
Septic Tank Repairs	100.00	100.00	——	——
Incinerator Fuel and Light	1,000.00	1,000.00	——	——
Electric Lighting—Streets and Alleys, and White Way	14,500.00	14,000.00	——	——
Interest on Warrants	2,284.00	2,170.00	——	——
Contingent Fund	5,000.00	4,036.00	——	——
Detention Hospital Insurance		432.75	——	——
Library Insurance		281.00	——	——
Park Insurance		104.25	——	——
	$35,944.00	$32,224.00	——	——
SUMMARY				
Water Department	$57,296.00	$52,541.00	——	——
Public Buildings and Utilities	$35,944.00	$32,224.00	——	——
	$93,240.00	$84,765.00	——	——

12. In the following find (*a*) the rate per cent for each item for 1923; (*b*) the increase or decrease in each item for the two years; (*c*) the per cents of increase or decrease based on the figures for 1922.

Explanation.—Net sales is the base, or 100 per cent. Each item is a percentage of net sales. The percentage divided by the base equals the rate.

DETERMINATION OF CAUSES OF VARIATION IN PROFITS OF THE ST. PAUL COMPANY

	1922		1923		Increase or Decrease*	
	Amount	Per Cent	Amount	Per Cent		Per Cent
Sales (Net)	$200,000	100.00	$250,000			
Cost of Goods Sold	150,000	75.00	180,000			
Gross Profit	50,000	25.00	70,000			
Selling Expenses	12,000	6.00	16,000			
Gen. Adm. Expenses	17,000	8.50	20,000			
Total Expenses	29,000	14.50	36,000			
Net Profit from Operations	21,000	10.50	34,000			
Financial Income	3,000	1.50	1,200			
Net Profit	24,000	12.00	35,200			

13. Complete the following statement analysis:

STATEMENT ANALYSIS OF THE *X* MANUFACTURING COMPANY
STATEMENT OF OPERATIONS YEAR ENDED DECEMBER 31, 1921

			Per Cent of Cost	*Per Cent of Net Sales*
Gross Sales		$186,444.29		xxxx
Less: Discounts Allowed	$ 3,529.46			
Freight Out	5,953.55	9,483.01		xxxx
Net Sales		176,961.28		xxxx
Cost of Sales				
Inventory, January 1, 1921	1,626.27			
Purchases	53,965.51			
	55,591.78			
Less Inventory Dec. 31, 1921	2,310.00			
	53,281.78		xxxx	
Factory Direct Expense				
Labor	$30,604.15		xxxx	
Factory Supplies and Mdse.	3,838.99		xxxx	
Containers and Cartons	26,425.51		xxxx	
Total Direct Expense	60,868.65			
Factory Overhead Expense				
Power and Light	$3,055.37			
Coal	1,886.80			
Insurance	2,602.72			
Taxes—Real Estate	1,946.71			
Repairs	149.57			
Oil	45.50			
Water	31.26			
Towels	38.55			
Management	4,000.00			
Depreciation—Buildings	3,160.16			
Depreciation—Machinery and Equipment	3,140.29	20,056.93	xxxx	
Cost of Goods Manufactured		134,207.36	xxxx	
Add—Invt. Finished Goods, Jan. 1, 1921		12,966.98		
		147,174.34		
Deduct—Invt. Finished Goods, December 31, 1921		11,196.68		
Cost of Goods Sold		135,977.66		xxxx
Gross Profit		40,983.62		xxxx
Selling Expenses		21,841.91		xxxx
Net Profit on Sales		19,141.71		xxxx
Management and Office Expenses		8,410.62		xxxx
Net Profit		$ 10,731.09		xxxx

14. In the following statement of earnings and expenses calculate (*a*) net operating revenue per mile; (*b*) net revenue; (*c*) per cents operating expenses are of operating revenue; (*d*) per cents taxes are of operating revenue; (*e*) per cents net revenue is of operating revenue.

NORTHERN PACIFIC RAILWAY

EARNINGS AND EXPENSES PER MILE OPERATED

	1917	*1918*	*1919*	*1920*	*1921*
Operating Revenues per Mile	13,526.37	15,594.28	15,282.27	16,996.50	14,199.10
Operating Expenses per Mile	8,171.39	10,857.13	11,934.71	15,177.88	11,659.73
Net Operating Revenue per Mile					
Taxes per Mile	1,059.52	1,236.01	1,365.42	1,519.34	1,353.87
Net Revenue					

RATIOS

	1917	*1918*	*1919*	*1920*	*1921*
Operating Expenses to Operating Revenue	——	——	——	——	——
Taxes to Operating Revenue	——	——	——	——	——
Net Revenue to Operating Revenue per Mile	——	——	——	——	——

15. In the following statement of gross profit, determine the gross profit for each department, the per cent of gross profit on sales, and the total.

Instructions.—To verify the additions and subtractions, foot the sales and cost-of-goods-sold columns and take their difference. This difference should equal the total of the gross profit column. The gross profit of each department divided by the sales of each department equals the per cent of gross profit on sales.

GROSS PROFIT OF A DEPARTMENT STORE

Dept. No.	*Sales*	*Cost of Goods Sold*	*Gross Profit*	*Per Cent Gross Profit on Sales*
A	$3,865.10	$3,266.85	——	——
B	5,875.65	5,416.92	——	——
C	4,876.18	3,925.67	——	——
D	2,397.56	2,107.12	——	——
E	1,486.89	987.34	——	——
F	6,567.45	5,756.23	——	——
G	4,543.23	3,986.75	——	——
H	4,765.98	4,235.44	——	——
I	2,342.24	1,835.55	——	——
J	4,977.55	4,671.10	——	——
K	3,222.66	2,765.45	——	——
L	5,045.54	4,552.34	——	——
Total				

16. Find the per cent that the value of output is of the invested capital for each year in the tabulation below:

RATIO OF OUTPUT TO INVESTED CAPITAL

Manufactured in Southern States Traversed by Southern Railway	*Capital Invested*	*Value of Output*
1904	$ 975,956,000	$1,036,428,000
1909	1,461,143,000	1,455,927,000
1914	1,778,556,000	1,727,393,000
1919	3,458,769,000	4,517,728,000

(The Southern States included in this compilation are Virginia, North Carolina, South Carolina, Georgia, Florida, Alabama, Mississippi, Tennessee and Kentucky.)

17. Find what per cent the net earnings are of the gross earnings for each year in the exhibit below:

PER CENT OF NET EARNINGS TO GROSS EARNINGS

NEW YORK STATE GAS AND ELECTRIC CORPORATION, ITHACA, NEW YORK, FOR THE YEARS ENDED DECEMBER 31ST

EARNINGS:	*1917*	*1918*	*1919*	*1920*	*1921*
Gross Earnings	$573,436	$631,740	$746,896	$903,855	$1,066,306
Expenses and Taxes (except Federal Taxes)...	396,024	470,233	475,256	627,110	707,046
Net Earnings	$177,412	$161,507	$271,640	$276,745	$ 359,260
	%	%	%	%	%

18. In the following tabulation, pro-rate $17,545.16 over the five departments on the basis of sales.

Explanation.—Assume the expenses, $17,545.16, are to be apportioned to the five departments in the same proportion that the sales of each department are to the total sales. To pro-rate the $17,545.16, divide it by the total sales; this gives the per cent of total expense to total sales. Assuming each department will bear expenses in proportion to its sales, the sales of each department multiplied by this rate will give the expense for each department.

PRO-RATIO AND PERCENTAGE

Department	*Sales*	*Expenses*	*Per Cent*
A	$12,421.50	——	——
B	13,345.65	——	——
C	17,638.58	——	——
D	14,363.21	——	——
E	85,695.19	——	——
		$17,545.16	100.00%

Another method would be to find what per cent of the total sales is made by each department, and since each department is to bear expenses in proportion to its sales, each department's rate per cent multiplied by $17,545.16 would give the departmental expense burden.

Chapter 5

MERCHANDISE CONTROL

Merchandise turnover.—The number of times the inventory is contained in the cost of sales is the merchandise turnover.

The final inventory should not be used in computing turnover unless it represents a normal inventory for the fiscal period, or is the first inventory that has been taken.

If a perpetual inventory system is in use, the monthly inventories should be added to the inventory at the beginning of the period and the sum divided by the number of months in the fiscal period plus one. In a year there would be thirteen inventories, the one at January 1, and the twelve inventories at the end of months. When a perpetual inventory is not used, add the inventory at the beginning of the fiscal period to the one at the close of the period; then divide by two. The quotient will be the approximate average inventory for the period.

An approximate inventory at the end of any period may be obtained by dividing the sales for the period by 100 per cent plus the per cent of gross profit based on cost, and deducting the quotient from the goods available for sale. A more complete discussion of the gross profit test is given in Chapter 6.

Cost of Sales ÷ Average Inventory at Cost = Rate of Turnover.

Illustrations.—A department store found the average inventory of Department *A* for the fiscal period was $30,000. The cost of sales for the same period was found to be $120,000.

$120,000 ÷ $30,000 = 4, rate of turnover.

Assuming that this department had goods available for sale worth $150,000, and sales amounting to $144,000, with an average mark-on of 20%, and using the gross profit test

to determine the average inventory, the solution would be as follows:

$144,000, sales ÷ 120% = $120,000, cost of sales.
$150,000, total cost of goods — $120,000 = $30,000, inventory.
$120,000, cost of sales ÷ $30,000, inventory = 4, rate of turnover.

Number of turnovers.—The number of turnovers varies in different lines of business. Records show turnovers varying from one to twenty, depending on the kind of business. It is possible to make a larger profit by several turnovers at a small mark-on than by one or two turnovers at a large mark-on. If capital is limited, frequent turnovers will produce a profit equal to that produced by turning a greater capital fewer times per year. If a merchant can turn $1 eight times in the course of a year, he has used one eighth of the capital that would be required if the rate of turnover were 1, and in addition has made as much or even more profit.

Illustration (*a*).—A merchant had a mark-on of 50% with a turnover of 1. He found that by using a mark-on of 30% he had a turnover of 2. If his former sales were $300,000 annually, how much were his gross profits increased provided he continued to use the same investment in merchandise?

$300,000 ÷ 150% = $200,000, cost of sales.
$300,000 — $200,000 = $100,000, gross profit.

Under the new policy he turns the $200,000 twice, an equivalent of $400,000 annually.

$400,000 at 30% = $120,000, profit.
$120,000 — $100,000 = $20,000, increased profits due to lowering mark-on and increasing rate of turnover.

Illustration (*b*).—How much of an investment in merchandise would be required under the new policy to make the same amount of profit that was made under the old policy?
$100,000 ÷ 30% = $333,333.33, cost of goods sold to make $100,000 profit. Since this volume is used in two turn-

overs, the amount of capital required would be $333,333.33 ÷ 2 or $166,666.67.

The merchant could make the same amount of profit, therefore, with an investment $33,333.33 smaller than under his old policy.

Problems.

1. A mark-on of 30% results in two turnovers costing $20,000 each. If expenses of conducting the business are $5,000, what is the net profit?

2. A mark-on of 20% results in three turnovers costing $20,000 each. If expenses remain at $5,000, what is the net profit?

3. The cost of sales in Department *B* was $52,000. The average inventory was $13,000. What was the rate of turnover?

4. A merchant's sales amounted to $30,000. His average inventory was $10,000 and the average mark-on was 50%. Find the rate of turnover.

5. Commodity X, with a mark-on of 40%, had a turnover of 2. With a mark-on of 30% it had a turnover of 3. If prior sales were $28,000, what are the present sales and the increase in profit?

6. A mark-on of 35% results in a turnover of 2, and sales amounting to $405,000. A mark-on of 20% results in a turnover of 4. How much less capital under the latter plan would make as much profit as under the former plan?

7. On January 1, 1922, a concern dealing in a single commodity had an inventory of merchandise which cost $20,000.00. The goods were marked to sell at 125% of cost and all subsequent purchases during the six months ending June 30, 1922, were marked at the same rate.

The selling price of the inventory at June 30, 1922, was $24,000.00. Purchases and sales by months were:

	Purchases (*Cost*)	*Sales* (*Selling Price*)
January	$ 8,000.00	$ 9,000.00
February	9,000.00	9,500.00
March	14,000.00	12,000.00
April	16,000.00	18,000.00
May	13,000.00	22,000.00
June	10,000.00	18,000.00

Compute estimated inventories at cost price at the end of each of the six months.

Compute the rate of turnover for the six months' period using (*a*) the January 1st and June 30th inventories; (*b*) all of the inventories.

State which method gives the more accurate results.—*American Institute Examination.*

Per cent of mark-down to net cost.—If an item costs $1 and is marked $1.25, in order to sell it for cost the price must be reduced 25 cents. The marked price is the base when reducing prices. 25 cents is ⅕ of $1.25. ⅕ = 20%.

An item costs $1 and is marked $1.50. 50 cents reduction is ⅓ of $1.50 or 33⅓%.

Problems.

Calculate the per cent of mark-down for the following items:

Items	*Cost*	*Marked Price*	*Per cent of Mark-Down to Produce Cost*
A	2.00	2.25	
B	1.00	1.50	
C	.20	.25	
D	.60	.75	
E	10.00	15.00	
F	2.50	2.75	
G	.40	.50	
H	.09	.12	
I	.03	.05	
J	3.50	7.00	
K	50.00	75.00	
L	24.00	30.00	

Computation of inventory by the retail method.—The need for frequent inventories has led many department stores to adopt the "retail method" of computing inventories. The accuracy of the inventory by this method depends upon the care exercised in recording the mark-ups and the mark-downs of merchandise prices, and the classification of merchandise into departments.

Procedure (by departments).—(*a*) Extend the inventory at the beginning of the month at both cost and retail values.

(*b*) Add purchases extended at both cost and retail, less

whatever adjustments may be necessary for returned purchases.

(*c*) To the total of (*a*) and (*b*) add the mark-ups to the retail value.

(*d*) From the total retail value deduct the total cost value and divide the difference by the total retail value to obtain the per cent of mark-on.

(*e*) Deduct the mark-downs from the total retail value.

(*f*) Deduct the sales at retail, and the remainder is the book inventory at retail price.

(*g*) Multiply the book inventory at retail price by the per cent obtained in (*d*).

(*h*) Subtract the product obtained in (*g*) from the book inventory at retail price and the remainder represents the inventory at cost value.

Example.—The records for Department *A* disclose the following information:

Inventory at beginning of month:	
Cost	$30,000
Retail	43,000
Purchases:	
Cost	46,000
Retail	55,000
Returned Purchases:	
Cost	1,000
Retail	1,500
Mark-ups	3,500
Mark-downs	5,000
Sales at Retail	71,000

Compute the cost of the book inventory.

Solution.

	Cost	*Retail*
Inventory	$30,000	$43,000
Purchases	46,000	55,000
	76,000	98,000
Deduct: Returned Purchases	1,000	1,500
	75,000	96,500
Mark-ups		3,500
		100,000

$100,000 − $75,000 = $25,000.
$25,000 ÷ $100,000 = 25%.

Mark-downs	5,000
	95,000
Sales at retail	71,000
End of month inventory of retail value	24,000

$24,000 × 25% = $6,000.
$24,000 — $6,000 = $18,000, the cost value of the inventory.

Problems.

1. From the records kept for Department *B*, the following information is obtained:

Inventory at beginning of month:	
Cost	$10,000.00
Retail	20,000.00
Purchases:	
Cost	$30,000.00
Retail	42,000.00
Returned Purchases:	
Cost	$ 500.00
Retail	600.00
Mark-ups	$ 2,000.00
Mark-downs	$ 1,000.00
Sales at retail	$50,000.00

Calculate the cost of the book inventory at the end of the month by the retail method.

2. In a certain department of a large dry-goods house the purchases for a year were $30,000.00. They were in the first place marked up for "selling" purposes to $45,000.00. Later additional mark-ups amounting to $2,000.00 were made and mark-downs were also recorded aggregating $5,000.00. At the end of the fiscal period there were found to be on hand goods of the marked selling value of $10,000.00. State how you would arrive at their inventory value for the purpose of closing the books and calculate the amount. Explain fully. —*American Institute Examination.*

Chapter 6

GROSS PROFIT TEST

Gross profit test.—Where the operations of a business have been fairly uniform for a period of years, it is advisable to test the gross profit shown by the Operating or Profit and Loss Statement.

Rate per cent of gross profit.—The gross profit test is based on the supposition that in normal times and under normal conditions, any business will produce approximately the same per cent of gross profit on sales in any one period of time as in any other period of time.

Procedure.—Statements of the gross profit and sales for each of several prior years should be obtained. The gross profit for each year divided by the sales for the same year gives the rate of gross profit for that year, *based on sales.* Eliminate any per cent that is abnormal. Add the remaining ones and divide by the number added. The quotient is the average per cent of gross profit of prior periods.

Cost of goods sold.—This average rate per cent applied to the sales for the current period will give the approximate gross profit for the current period. Deducting the approximate gross profit from the sales gives the approximate cost of goods sold. This may be reduced to a formula as follows:

Average for Prior Periods.

1. Sales — Cost of sales = Gross profit.
2. Gross profit ÷ Sales = Per cent of gross profit (based on sales).

Application to Current Period.

3. Sales × Per cent of gross profit (prior periods) = Estimated gross profit.
4. Sales — Estimated gross profit (current period) = Estimated cost of sales.

Example.

	Sales	*Cost of Sales*	*Gross Profit*
First period	$400,000	$300,000	$100,000
Second "	450,000	340,000	110,000
Third "	350,000	260,000	90,000
Fourth "	100,000		

What was the cost of sales during the last period?

Solution.

Average for Prior Periods.

First period,	$100,000 ÷	$400,000 =	25%.
Second "	110,000 ÷	450,000 =	24.44%.
Third "	90,000 ÷	350,000 =	25.71%.
			75.15%.

75% ÷ 3 = 25%, average gross profit.

Application to Current Period.

$100,000 × 25% = $25,000, estimated gross profit.
$100,000 — $25,000 = $75,000, estimated cost of sales.

Rate per cent of cost of sales.—If the profit has been based on cost price instead of on selling price, the Cost of Sales may be tested by the following computations:

Average for Prior Periods.

1. Sales — Cost of sales = Gross profit.
2. Gross profit ÷ Cost of sales = Per cent of gross profit (based on cost of sales).

Application to Current Period.

3. Sales ÷ (100% + Per cent of Gross profit) = Cost of sales.

Example.

	Sales	*Cost of Sales*
First period	$400,000	$300,000
Second "	450,000	340,000
Third "	350,000	260,000
Fourth "	100,000	

What was the cost of sales for the last period?

Solution.

Average for Prior Periods.

First period,	$100,000 ÷	$300,000 =	33.33%.
Second "	110,000 ÷	340,000 =	32.35%.
Third "	90,000 ÷	260,000 =	34.61%.
			100.29%

100% ÷ 3 = 33⅓%, average gross profit.

Application to Current Period.

$100,000 ÷ 1.33⅓ (1. + .33⅓) = $25,000, gross profit.
$100,000 — $25,000 = $75,000, cost of sales, current period.

It follows that if the cost of sales can be found, other things being given, any element (inventory at beginning of period, purchases, closing inventory, etc.) which goes to make up the cost of sales can be found, providing the other elements of the costs are given.

Of course, no test such as this will be absolutely accurate. On large amounts there may be a variation from what would have been the accurate result, but if no radical difference exists there is no reason to believe the results to be materially wrong.

Uses.—The gross-profit test is useful in two very important ways: first, to prove inventories; and second, to compute the approximate inventory in case of a fire or in case of the necessity to purchase at a date when a physical inventory cannot be taken.

Illustration.—Assume that the average gross profit for the past five years has been 40 per cent of sales, and that in auditing the books it is found that the inventory, taken prior to beginning the audit and valued at $100,000, seemed smaller than usual, while the previous inventory and purchases amounted to $400,000. The sales were $400,000.

Comparative statement.—Set up a comparative statement. Show both the average and the current percentages and results. As the profit in prior years has been 40% of sales, the cost of goods sold has been 60% of sales. 60% of $400,000, sales = $240,000, estimated cost of sales. The statement would be as follows:

COMPARATIVE STATEMENT

		Actual Amounts	*Current Per Cent*	*Test Amounts*	*Average Per Cent*
Sales		$400,000	100	$400,000	100
Cost of Sales:					
Old Inv. and Purchases	$400,000				
Less New Inventory	100,000	300,000	75	240,000	60
Gross Profit		$100,000	25	$160,000	40

If the sales are correct, the cost of sales is $60,000 too high, unless the rate has really changed. The accountant should determine the reason for this marked difference.

Fire losses.—In fire losses, insurance companies are generally willing to settle on the basis of a gross profit valuation of inventories of goods on hand at the time of the fire.

Example.—The Insurance Company agrees to pay the insured for his loss based on the following facts:

Average gross profit for 4 years, 40% on sales.
Sales for this period to date of fire, $50,000.
Goods available for sale, $300,000.

Solution.

Sales, $50,000 × 40%, rate of gross profit = $20,000, gross profit.

Sales, $50,000 — Gross profit, $20,000 = $30,000, cost of goods sold.

Goods available for sale, $300,000 — Cost of goods sold, $30,000 = $270,000, approximate inventory at date of fire.

Problems.

1. From the figures below calculate the per cent of gross profit for each year, and by means of the average per cent of gross profit, calculate the inventory at the middle of the year 1922.

Date	*Sales*	*Purchases*	*Old Inv.*	*New Inv.*	*Per Cent Gross Profit*
1918	$100,000	$200,000		$140,000	——
1919	150,000	150,000	$140,000	203,000	——
1920	140,000	110,000	203,000	213,000	——
1921	220,000	140,000	213,000	200,000	——
1922 (6 mo.)	110,000	105,000	200,000	——	——

2. From the following facts find the inventory as of December 31, 1922:

Inventory January 15, 1923, $15,378.50.

Sales December 31 to January 15, $1,890.00.

$600 of the above sales shipped and invoiced before December 31.

Purchases December 31, to January 15 at cost, $1,356.50.

Average gross profit is 25% of cost.

3. The average gross profits of the *X* Company for the past three years have been 50% of the sales. During 1921

the sales amounted to $120,000. Goods were purchased to the amount of $100,000. Purchases were returned for the period to the amount of $5,000. Freight paid on purchases was $6,000. The inventory at the beginning of the period amounted to $40,000. Current market prices are 10% above the purchase prices for the year. Find the cost of replacing the goods at the end of the year.

4. The board of directors of the *X Y Z* Company removed their manager on April 30, 1918, on the general suspicion that his books misrepresented the true financial condition of the business. Prepare a statement showing the nature and the probable extent of the misrepresentation, also an approximate statement of income and profit and loss for the four months ending April 30, 1918, and a balance sheet as of April 30, 1918.

The following is a trial balance taken from the books, April 30, 1918:

Capital Stock		$ 75,000.00
Fixtures	$ 10,000.00	
Inventory, January 1, 1918	128,600.00	
Cash	15,450.00	
Accounts Receivable	24,600.00	
Accounts Payable		39,000.00
Loans Payable		10,000.00
Sales		51,000.00
Purchases	40,700.00	
Salaries, Salesmen	2,200.00	
Advertising	1,650.00	
Salaries, Office	1,100.00	
Rent	400.00	
Interest	200.00	
Insurance, Jan. 1 to December 31, 1918	999.00	
Stationery and Printing	105.00	
Reserve for Depreciation of Fixtures		2,710.00
Surplus, January 1, 1918		48,294.00
	$226,004.00	$226,004.00

An analysis of the Purchases and Sales accounts revealed the following: purchases, year 1915, $122,000; sales, year 1915, $153,750; inventory, January 1, 1915, $101,000; purchases, year 1916, $123,000; sales year 1916, $153,170; inventory, January 1, 1916, $100,000; purchases, year 1917, $121,000; sales, year 1917, $154,722; inventory, January 1, 1917, $102,000.—*C. P. A. Examination.*

5. The books of a concern recently burned out contained evidence of purchases, including inventory of $200,000.00, and sales of $40,800.00 since the last closing. Upon investigation, however, the auditor ascertained that a sale of merchandise had been made just prior to the fire, and not recorded in the books, at an advance of two-fifths over cost less a 10% cash discount, on which the profit was $31,928.00. The past history of the business indicated an average gross profit of 50% on cost of goods sold.

(*a*) What amount should be claimed as fire loss?

(*b*) What rate of gross profit do the transactions finally yield?—*American Institute Examination.*

Chapter 7

FINANCIAL AND OPERATING RATIOS

Mathematics of credits.—Credit men and bankers analyze the balance sheet and the profit and loss statement of the individual, partnership, or corporation seeking credit or making application for a loan. The following analyses are usually made:

Ratio of Current Assets to Current Liabilities.
" " Accounts Receivable to Sales (Accounts Receivable Turnover).
" " Cost of Sales to Inventory (Merchandise Turnover).
" " Gross Profit to Net Sales.
" " Selling Expenses to Net Sales.
" " Net Profit to Net Sales.
" " Net Operating Expenses to Net Sales.
" " Net Profit to Net Worth.

Merchandise turnover has been illustrated under the subject of Turnover. The methods of computing per cents on Net Sales, viz., Gross Profit, Selling Expenses, Net Profit, Net Operating Expenses, etc., were discussed in the chapter on Percentage.

Ratio of current assets to current liabilities.—The popular idea of the proper ratio which current assets should bear to current liabilities is 2 : 1, although some desire a different ratio, depending on the kind of business.

Current assets divided by current liabilities equals the number of times the current assets are greater than the current liabilities. If the quotient is 2, the current assets are in a 2 to 1 ratio.

Ratio of accounts receivable to sales.—The normal credit period, whether it be 30, 60, or 90 days, is compared with the

credit period obtained from the following formula, as a means of judging the efficiency of the collection department:

$$\frac{\text{Accounts receivable at end of fiscal period}}{\text{Sales for fiscal period}} \times \text{Months in fiscal period} = \text{Average credit period.}$$

Example.—The Accounts Receivable account showed $125,000 worth of outstanding accounts at the close of the fiscal period. The sales for the fiscal period of 12 months amounted to $1,000,000, on which the average term of credit granted at time of sale was 30 days. What is the average credit period of the outstanding accounts?

Solution.

$$\frac{\$125{,}000}{\$1{,}000{,}000} \times 12 = 1\tfrac{1}{2} \text{ months, average credit period.}$$

This would indicate that the average credit period of Accounts Receivable outstanding is 45 days, although the term of credit granted was 30 days.

Problems.

1. The Philadelphia Company's consolidated balance sheet for December 31, 1922, shows current assets, $2,735,618.29 and current liabilities, $1,223,948.75. What is the ratio?

2. The Equitable Sales Company's balance sheet of December 31, 1922, shows current assets of $451,582.92, and current liabilities of $137,077.26. What is the ratio?

3. A company's sales for a year were $849,656.51. The balance sheet at the end of the same period shows Accounts Receivable, $315,279.66. If the average term of credit is 60 days, what condition is shown by the above facts?

4. A company's sales for a year are $876,578.01. The balance sheet at the end of the same period shows Accounts Receivable of $24,078.70. If the average term of credit is 30 days, what conclusion would you draw from the facts given?

Profits distribution.—The distribution of profits under the partnership type of business organization will be pre-

sented under the subject of partnership. The distribution of the profits of corporations will be illustrated in the following examples.

Example.—The Equitable Towing and Transportation Company had 2250 shares of common stock, par value $100, outstanding December 31, 1922, and declared a 2% dividend. Net earnings were $53,320.84. What amount of the year's profits, after the declaration of the dividend, remained in Surplus? What was the amount of the dividend?

Solution.

2250 shares at $100 = $225,000, capital stock.
2% of $225,000 = $4,500, dividend.
$53,320.84 — $4,500 = $48,820.84, amount of year's profits remaining in surplus.

Example.—A company had the following number of outstanding shares, each with a par value of $50:

Common	858,860 shares
Preferred 6% Cumulative	291,047 "
Preferred 5% Non-cumulative and Non-participating	28,849 "

The company declared a 6% dividend on the common stock in that year. What amount of profits was distributed to each class of stock? What amount of profits was necessary to cover dividends for that year?

Solution.

	Capital	*Dividends*
858,860 shares Common at $50	$42,943,000	
6% of $42,943,000		$2,576,580.00
291,047 shares Preferred 6% Cumulative at $50	14,552,350	
6% of $14,552,350		873,141.00
28,849 shares Preferred 5% Non-cumulative at $50	1,442,450	
5% of $1,442,450		72,122.50
Profits necessary to cover dividends		$3,521,843.50

Problems.

1. The Equitable Auto Company had 1650 shares of stock outstanding December 31, 1922. The par value of each share was $100. The Surplus Statement shows "Dividend on Capital Stock, $16,500." What per cent of dividend was paid?

2. A company has 182,260 shares of common stock, and 57,633 shares of 7% cumulative preferred stock outstanding. The par value of each class is $100. If the company declares a 6% dividend on common stock, what is the total amount of dividends for the year?

3. A company paid $246,800 in dividends on outstanding common capital stock of $6,170,000. What was the rate?

4. A company had net earnings for 1921 amounting to $2,850,460.03. It paid from this 8% dividends on $12,379,850 preferred stock outstanding. If the preferred stock is non-participating, what per cent was earned for the common stockholders if there was $10,600,000 of common stock outstanding?

Chapter 8

COMMERCIAL DISCOUNTS

Definitions.—Cash or time discount is a deduction for immediate payment, or for payment within a definite time. The deduction is a certain per cent of the bill if paid within a fixed time.

The expression, "Terms, 3 mos. net, 2/30, 5 cash," on a billhead means the goods are bought on 3 months' time, but if paid for within 30 days, a discount of 2% will be allowed, or 5% will be allowed for cash.

Mercantile or trade discounts are reductions from list prices, or from the amount of a bill of goods. These discounts enable the merchants to catalogue their merchandise and sell to both wholesalers and retailers, or to advertise their goods at certain prices. The wholesaler receives a different discount sheet from the one received by the retailer. Revised discount sheets are issued as prices fluctuate, whereas the same catalogue may be used for a year or more because of the fixed list prices.

Invoice Price = Base.
Per Cent of Discount = Rate.
Discount = Percentage.

Several discounts are sometimes given. These are known as chain discounts, and the several discounts are referred to as a series of discounts.

The order in which the discounts are deducted will not affect the result; thus, a selling price stated as list price "less 10%, 20%, and 5%" is the same as a selling price stated "less 5%, 20%, and 10%" as is shown in the following solution using $100 as the base:

$100 × .10 = $10
$100 — $10 = $90
$90 × .20 = $18
$90 — $18 = $72
$72 × .05 = $ 3.60
$72 — $ 3.60 = $68.40
$100 — $68.40 = $31.60, the total discount.

$100 × .05 = $ 5
$100 — $ 5 = $95
$95 × .20 = $19
$95 — $19 = $76
$76 × .10 = $ 7.60
$76 — $ 7.60 = $68.40
$100 — $68.40 = $31.60, the total discount.

Several discounts are *not* equal to their sum. The sum of the rates in the preceding illustration is 35%. 35 per cent of $100 is $35, whereas the correct discount is $31.60.

Single discount equivalent to a series (First method).—To find the single discount that is equivalent to a series of discounts, subtract one of the discounts from 100 per cent. Use the remainder as a new base. Multiply it by the second discount and deduct the product. Use this remainder as a new base. Compute each discount successively, proceeding as before. The difference between 100 per cent and the last result will be the single discount.

Example.—What single discount is equivalent to a series of discounts of 20%, 10%, and 8⅓%?

Solution.

100% — 20% = 80%
80% × 10% = 8%
80% — 8% = 72%
72% × 8⅓% = 6%
72% — 6% = 66%
100% — 66% = 34%

Explanation.—100% represents invoice price. 20%, or ⅕ of 100% equals 20%, which subtracted from 100% leaves 80%; 10% or $\frac{1}{10}$ of 80% equals 8%, which subtracted from 80% leaves 72%; 8⅓%, or $\frac{1}{12}$ of 72% equals 6%, which subtracted from 72% leaves 66%. 100%, invoice price, less 66%, selling price, leaves 34%, single discount.

(Second method).—

Procedure.—Subtract each single discount from 100 per cent, and find the product of the remainders. Subtract the final product from 100 per cent.

Example.—What single discount is equivalent to the series 20%, 10%, and 5%?

Solution.

100%	100%	100%
20%	10%	5%
80%	90%	95%

.80 × .90 × .95 = .684 or 68.4%
100% — 68.4% = 31.6%, the single equivalent

A short method.—To find the single discount that is equivalent to any two discounts: From the sum of the discounts take their product.

Example.—What single discount is equivalent to discounts of 20% and 20%?

Solution.

20% + 20% = 40%
20% of 20% = 4%
40% — 4% = 36%

To find the net price.—To find the net price, the list price and discounts being given: Reduce the discount series to a single discount, multiply the invoice price by this rate of discount, and deduct the discount from the invoice price.

Example.—What is the net price of an invoice of $600, less 30%, 20%, and 10%?

Solution.

100%	100%	100%
30%	20%	10%
70%	80%	90%

.70 × .80 × .90 = .504, or 50.4%.
100% — 50.4% = 49.6, single discount.
$600 × 49.6% = $297.60, discount.
$600 — $297.60 = $302.40, net price.

If the amount of discount is not desired, the net price may be found as follows:

$600 × 50.4% = $302.40.

Problems.

1. What discount is equivalent to discounts of 5% and 30%?

2. What discount is equivalent to discounts of 10% and 60%?

3. What discount is equivalent to discounts of 70% and 10%?

4. What discount is equivalent to discounts of 70% and 30%?

5. What discount is equivalent to discounts of 80% and 10%?

6. A salesman offers to sell $3,000 worth of groceries at discounts of 25%, 10%, and 10%. Another offers to sell the same quantity for a like amount, less 20%, 20%, and 5%. Which is the better offer, and by how much?

7. *X* bought $4,500 worth of merchandise at 20% and 25% off. He sold it for $4,500, less 15%, 10%, and 20%. How much did he gain or lose?

8. A bill of tinware is sold at the following discounts: $74.20 at 20% and 10%; $43.75 at 40% and 5%; $60 at 33⅓% and 10%; and $49.17 net. What is the total amount of the bill?

9. A bill of dry goods amounting to $914.37 is sold August 19, on the following terms: "2/10, 1/30, n/60." How much would settle the bill September 18? How much August 27?

10. Of a bill of hardware, $61.51 is sold at a discount of 60% and 5%; $18.75 at a discount of 10%; $16.86 at a discount of 12½%; $44.25 at a discount of 40% and 5%; $29.60 at a discount of 40%, 12½%, and 10%; $28.04 at a discount of 55%; $16 at a discount of 65%, 10%, and 10%; $18.50 at a discount of 50%; $19.75 at a discount of 20%; $18.50 at a discount of 15%; $307.55 at a discount of 75% and 12½%; $36.61 at a discount of 60% and 10%; and $218.25 net. What is the total net amount of the bill?

Chapter 9

SIMPLE INTEREST

Definition.—As commonly defined, interest is a payment for the use of money. This payment for a certain sum depends upon the rate per cent charged and the length of time for which the sum is loaned. The sum for which the charge is made is the principal. The rate is the number of hundredths of the principal that is taken, and is usually expressed as a per cent. The principal plus the interest is called the amount.

Short methods of calculating.—There are a great many methods of computing interest, each of them possessing more or less merit. With the accountant it should not be a question of how many ways there are in which he can solve a problem in interest, but rather how accurately and quickly he can do it.

The computation of the product of principal, rate, and time is the shortest method, when the time is full years or fractional parts of a year, such as ½, ⅓, ¼, any number of 10ths, etc.; otherwise the operation may be shortened by taking advantage of aliquot parts, simple rates, cancellation, etc.

The following principles and methods of computing interest give short cuts for calculating interest when the rate is 6 per cent.

Sixty-day method.—To find the interest at 6% for:

6 days, point off 3 additional places to the left of the decimal point in the principal.

60 days, point off 2 additional places to the left of the decimal point in the principal.

600 days, point off 1 additional place to the left of the decimal point in the principal.

Example.—Find the interest on $256.75 for 6 days at 6%.

Solution.

Pointing off 3 places to the left in the principal gives .256, or 26 cents.

Example.—Find the interest on $345.65 for 36 days at 6%.

Solution.

Point off 3 additional decimal places in the principal and multiply by 6. The answer is $2.07.

For rates other than 6%, see adjustments on page 62.

Problems.

Find the interest at 6% on:

$175.00	for	60	days		
17.50	"	54	"		
275.00	"	24	"		
47.56	"	36	"		
75.55	"	12	"		
$ 83.12	for	18	days		
896.30	"	6	"		
12.70	"	54	"		
87.97	"	30	"		
6.75	"	24	"		

Method using aliquot parts.—

Example.—Find the interest on $275.84 for 124 days at 6%.

Solution.

$2	75.84 =	Interest	for	60	days
2	75.84 =	"	"	60	"
	18.38 =	"	"	4	"
5	70.06 =	"	"	124	days

Explanation.—Pointing off 2 decimals gives the interest for 60 days. Double this to find the interest for 120 days. Four days' interest is $\frac{1}{15}$ of 60 days' interest. The sum, $5.70, is the interest for 124 days.

Example.—Find the interest on $754.90 for 137 days at 6%.

Solution.

$7	54.90 =	Interest	for	60	days
7	54.90 =	"	"	60	"
1	50.98 =	"	"	12	"
	62.90 =	"	"	5	"
17	23.68 =	"	"	137	days

Explanation.—Point off 2 decimal places to find the interest for 60 days. Double this to find the interest for 120 days. Twelve days is $\frac{1}{5}$ of 60 days; therefore, the interest

for 12 days is ⅕ of 60 days' interest, or $1.5098. Five days is 1/12 of 60 days, and the interest is 1/12 of $7.5490, or $.629. The sum, $17.24, is the interest for 137 days.

A little practice will enable one to resolve any number of days into 6- or 60-day periods, and easy fractions thereof.

Example.—Find the interest on $247.64 for 8 days at 6%.

Solution.

$	247.64	= Interest for 6 days
	082.54	= " " 2 "
	330.18	= " " 8 days

Explanation.—Point off 3 decimals for 6 days. Two days equals ⅓ of 6 days. The answer is 33 cents.

For rates other than 6% see adjustments on page 62.

Problems.

Find the interest at 6% on:

$249.82	for	9	days		$274.19	for	34	days
384.07	"	8	"		15.12	"	19	"
309.15	"	27	"		39.84	"	17	"
407.24	"	39	"		29.81	"	2	"
84.29	"	40	"		1756.38	"	21	"

The cancellation method.—The cancellation method may be used to advantage on many interest calculations, especially those having odd rates.

Example.—Find the interest on $750 for 45 days at 5%.

Solution.

$$\frac{\cancel{750}^{125} \times \cancel{15}^{15} \times \cancel{.05}^{.01}}{\cancel{12}_{4} \times \cancel{30}_{6}} = \frac{18.75}{4} = 4.687 \text{ or } \$4.69.$$

Note.—Writing 12 times 30 instead of 360 days below the line facilitates cancellation.

Problems.

Find interest, by the cancellation method, on:

$345.00	for	12	days	at 2%	$175.40	for	34	days	at 6%
234.00	"	15	"	" 4%	785.56	"	56	"	" 5%
175.50	"	16	"	" 5%	902.25	"	79	"	" 5%
987.00	"	85	"	" 4%	416.16	"	91	"	" 6%
126.56	"	74	"	" 6%	175.00	"	102	"	" 4%

Example.—Find the interest on $345.75 for 90 days at 4½%.

Solution.

$$\frac{345.75 \times 90 \times .09}{\frac{1}{2} \times 30 \times 2} = \frac{31.1175}{8} = \$3.889 \text{ or } \$3.89.$$

Explanation.—Convert 4½ into an improper fraction, ⁹⁄₂, and write as above indicated; then proceed to cancel as before.

Problems.

Find interest, by the cancellation method, on:

1. $312.32 for 38 days at 4½%.
2. $275.25 for 45 days at 4¾%.
3. $568.30 for 40 days at 5½%.
4. $675.80 for 75 days at 4¼%.

Dollars-times-days method, 6 per cent.—This method is very rapid and is valuable for use in connection with calculating machines.

It is a modification of the cancellation method, where 6 per cent and 360 days are two of the factors, thus:

$$\frac{\$ \times \text{Days} \times 60}{360} \qquad 6000$$

Assume there are no other cancellable items. The number of dollars would be multiplied by the number of days and the product divided by 6000. Any product of dollars by days may be divided by 6000 by pointing off 3 decimals, and dividing the resultant number by 6.

Example.—Find the interest on $256.50 for 28 days at 6%.

Solution.

Multiply the dollars by the days, *point off 3 decimals in addition to the usual number, and divide by 6.*

$$\begin{array}{r} \$\ 256.50 \\ \underline{\quad\ 28} \\ 6)\ 7.182\ 00 \\ \hline 1.197 \end{array} \quad \text{or } \$1.20, \text{ interest.}$$

This method may be used for any rate by adding to or subtracting from the interest computed at 6%, the fractional

part thereof that the specified rate is greater or less than the 6% rate.

For 8% increase the interest by ⅓ of the amount computed at 6%
For 7% " " " " ⅙ " " " " " " "
For 5% decrease " " " ⅙ " " " " " " "
For 4% " " " " ⅓ " " " " " " "
For 4½% " " " " ¼ " " " " " " "

The above adjustments may be used with any of the 6 per cent methods in solutions where the rate is more or less than 6 per cent.

Problems.

Find the interest on the following:

1. $284.19 for 73 days at 5%.
2. 135.20 " 28 " " 6%.
3. 148.55 " 70 " " 4%.
4. $824.19 for 28 days at 4½%.
5. 294.36 " 17 " " 8%.
6. 128.95 " 56 " " 7%.

Interchanging principal and time.—It is possible to interchange the time and the principal in some problems, and by applying some one of the short methods in this chapter, shorten the calculation.

Example.—Find the interest on $6,000 for 31 days at 6%.

Solution.

Interchanging the principal and days, the problem is to find the interest on $3100 for 60 days. Apply the 6%, 60-day method, and the interest is found to be $31.

Problems.

Find the interest on the following:

1. $1,200 for 23 days at 6%.
2. 5,400 " 7 " " 6%.
3. 3,000 " 156 " " 6%.
4. $6,000 for 193 days at 6%.
5. 3,000 " 28 " " 6%.
6. 7,500 " 41 " " 6%.

Accumulation of simple interest.—Simple interest accumulates in like amount each period, if the principal and rate are unchanged.

Symbols.—Let i equal the rate of interest, n the number of periods, and P, the principal. Then accumulation of simple interest on any sum of money, for any number of periods, may be found as follows:

Example.—Find the simple interest on \$100 for 5 years at 6%.

Algebraic Formula.	*Arithmetical Substitution.*
$P\ (1 \times i\,n) =$ Interest.	$100\ (1 \times .06 \times 5) =$ Interest.

Solution.

1 × .06 = .06, interest on 1 for 1 year at 6%.
.06 × 5 = .30, interest on 1 for 5 years at 6%.
.30 × 100 = \$30, interest on \$100 for 5 years at 6%.

TABLE

1	2	3	4	5
End of Year	*Principal*	*Interest Due Each Year*	*Total Int. at End of Each Year*	*Sum Due at End of Year*
1.....	\$100.00	\$6.00	\$ 6.00	\$106.00
2.....	100.00	6.00	12.00	112.00
3.....	100.00	6.00	18.00	118.00
4.....	100.00	6.00	24.00	124.00
5.....	100.00	6.00	30.00	130.00

Problems.

Work out expressions and solutions, as shown above, for 1 and 2 below.

1. A man borrows \$300 for 9 years at 4%. What is the interest?

2. What is the amount of interest due at the end of 10 years, at 7%, on \$700?

3. Construct a table in columnar form similar to the above table (omitting Column 5), for \$300 invested at 8% for 5 years.

4. What is the interest accumulation on a debt of \$3275 for 8 years at 5% simple interest?

Simple amount.—The simple amount is found by adding the total simple interest to the principal. It is the amount due at the end of the stated time.

Example.—What is the amount of \$100 for 5 years at 6%?

Algebraic Formula.	*Arithmetical Substitution.*
$P + [P\ (1 \times i\,n)] =$ Amount.	$100 + [100\ (1 \times .06 \times 5)] =$ Amount.

Solution.

1 × .06 = .06, interest on 1 for 1 year at 6%.
.06 × 5 = .30, interest on 1 for 5 years at 6%.
100 × .30 = 30, interest on 100 for 5 years at 6%.
100 + 30 = $130, amount of $100 for 5 years at 6%.

In Column 5 of the preceding table, the simple amount is shown; therefore, the construction of a table is omitted.

Problems.

Construct expressions and obtain solutions for the following:

1. The amount of a $100 note due in 4 years at 5%.

2. The amount due in 6 years on $430 at 6%.

3. Amount due on a note dated August 1, 1916, for $890, with 4% interest, and due August 1, 1920. No interest has been paid.

Rate.—To find the *rate,* when the principal, interest, and time are given, divide the given interest by the interest on the principal at 1 per cent for the given time.

Example.—At what rate will $100 produce $24 in 4 years?

Algebraic Formula.

$$\frac{\text{Interest}}{P\,(1 \times i\,n)} = \text{Rate of interest.}$$

Arithmetical Substitution.

$$\frac{24}{100\,(1 \times .01 \times 4)} = \text{Rate of interest.}$$

Solution.

1 × .01 = .01, interest on 1 at 1% for 1 year.
.01 × 4 = .04, interest on 1 at 1% for 4 years.
100 × .04 = 4.00, interest on 100 at 1% for 4 years.
24 ÷ 4 = 6, or 6%, rate of interest.

Problems.

Work out formulas and solutions for the following:

	Principal	*Interest*	*Time*	*Rate*
1.	$ 600	$ 18	3 years	?
2.	$3,000	600	5 "	?
3.	$ 900	540	6 "	?
4.	$ 800	224	7 "	?
5.	$ 450	135	6 "	?

Time.—To find the *time,* when the principal, interest, and rate of interest are given, divide the interest by the interest at the given rate on the principal for one year.

Example.—In what time will $100 invested at 6% produce $24 interest?

Algebraic Formula.

$$\frac{1}{P\,(1 \times i\,n)} = \text{Time.}$$

Arithmetical Substitution.

$$\frac{24}{100\,(1 \times .06 \times 1)} = \text{Time.}$$

Solution.

1 × .06 = .06, interest on 1 at 6% for 1 year.
100 × .06 = 6, interest on 100 at 6% for 1 year.
24 ÷ 6 = 4, number of years.

Present worth.—The present worth of a debt, due at some future time, without interest, is the sum which must be invested now in order to obtain the specified amount at the end of the period.

To find the present worth, divide the principal by the simple amount of $1 for the given time.

Example.—What is the present worth of a note, without interest, for $100 due in 5 years, money being worth 6%?

Algebraic Formula.

$$P\left(\frac{1}{1 + (1 \times i\,n)}\right) = \text{Present Worth.}$$

Arithmetical Substitution.

$$100\left(\frac{1}{1 + (1 \times .06 \times 5)}\right) = \text{Present Worth.}$$

Solution.

1 × .06 = .06, interest for 1 year on 1.
.06 × 5 = .30, interest for 5 years on 1.
1 + .30 = 1.30, amount of 1 for 5 years.
1 ÷ 1.30 = .7692, present worth of 1 for 5 years.
100 × .7692 = $76.92, present worth of $100 for 5 years.

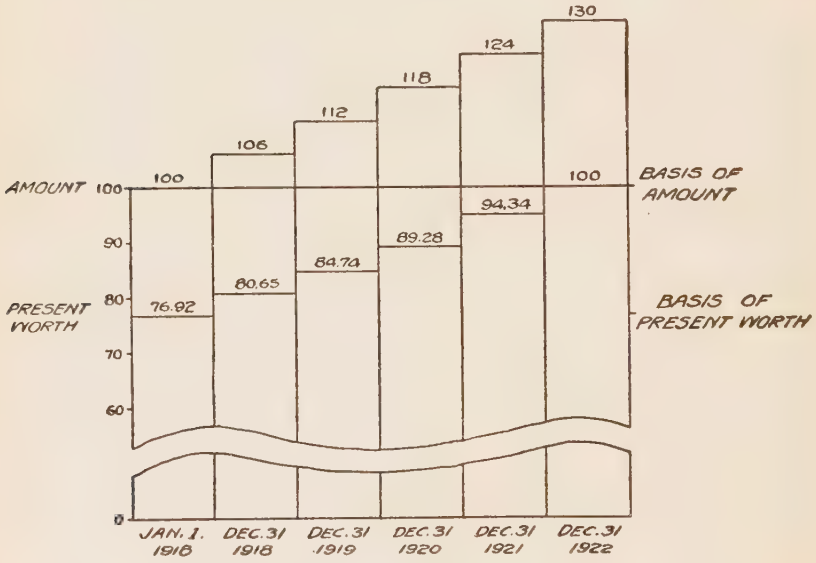
Verification.

$76.92 × .06 = $ 4.6154, interest for 1 year on present worth.
$ 4.6154 × 5 = 23.08, interest for 5 years on present worth.
$76.92 + 23.08 = $100, amount due in 5 years.

TABLE

1	2	3	4	5
Years	*Principal*	*Divided by Amount of $1*	*Equals Present Worth*	*Principal Minus Present Worth Equals Discount*
1.....	$100	$1.06	$94.34	$ 5.66
2.....	100	1.12	89.29	10.71
3.....	100	1.18	84.74	15.26
4.....	100	1.24	80.65	19.35
5.....	100	1.30	76.92	23.08

Comparison of simple amount and simple present worth.—The following comparative chart is presented to illustrate the accumulation of simple interest on a sum and on the present worth of the same sum:



The *amount* starts with the value of $100, and accumulates to $130 in 5 years. The *present worth* starts at $76.92 and accumulates to $100. The rate of interest is the same in each case, 6%.

Problems.

Work out formulas and solutions for 1 and 2.

1. What is the present value of a 6-year note, without interest, for $750, money being worth 5%?

2. A note of $3,500, without interest, is due in 5 years. What is the present value, money being worth 7%?

3. Construct a table in columnar form similar to the preceding table (omitting column 5), for 4 years at 8%, on a $1 principal.

4. Construct a comparative chart showing the difference in values of the simple amount and the simple present worth of $500 due in 8 years, interest at 5%.

True discount.—True discount is the difference between the face of a debt and its present worth.

Example.—Find the true discount on a debt of $100 due in 5 years, without interest, money being worth 6%.

Solution.

The present worth of the debt is the sum shown in the solution on page 65 under *present worth,* or $76.92.

$100, debt — $76.92, present worth = $23.08, true discount.

Refer to the Present Worth Table, Column 5, for the method of showing the true discount in columnar form.

Problems.

Show formulas and solutions for 1 and 2.

1. What is the true discount of a debt of $950 due in 3 years, money being worth 4%?

2. Find the difference between the present value and face value of a non-interest-bearing note of $700, due in 4 years, money being worth 8%.

3. Construct a table in columnar form, similar to the preceding table, for $1 at 4% for 4 years.

4. Find the difference between the true discount and the simple interest on $650 for 8 years at 5%.

Chapter 10

BANK DISCOUNT

Definition.—Bank discount is simple interest deducted in advance. It is computed on the value which a note will have at maturity. If the note does not bear interest, its face value is its value at maturity. If the note bears interest, the value at maturity is the face value plus interest on the face value for the period and at the rate specified in the note.

The time, in bank discount, is the period from the date of discount to the date of maturity of the note. The date of maturity of a note is the day on which it is due. If the time stated in a note is given in days, count the actual number of days to find the due date; but if the time in the note is given in months, the due date is found by counting the time by months.

Example.—A note due 30 days after January 31, 1921, will mature on March 2, 1921; but if due in 1 month, it will mature on the last day of the succeeding month, or February 28.

Counting time.—In counting time, the usual method is to count the first succeeding day as one day. To illustrate, if a note is given on January 15 for ten days, the 16th will be counted as one, the 17th as two, the 18th as three, and the 25th will be the tenth day.

Finding the difference between dates by use of a table.—

TABLE OF NUMBER OF DAYS FROM ANY DAY OF ANY MONTH TO CORRESPONDING DAY OF ANY MONTH NOT MORE THAN ONE YEAR LATER

	Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

Example.—A note due August 17, 1923, was discounted June 10, 1923. What was the term of discount?

Solution.

From the table, June 10 to August 10	=	61 days
August 10 to August 17	=	7 days
Total	=	68 days

Banks do not compute time uniformly, but the methods given here are in common use.

Proceeds.—The proceeds of a note is the difference between the amount due at maturity and the bank discount.

Procedure.—Compute the bank discount as simple interest on the amount due at maturity for the unexpired time (term of discount). Deduct the bank discount from the value of the note at maturity to obtain the proceeds.

To find bank discount and proceeds.—

Example 1.—Find the bank discount and proceeds if a non-interest bearing note for $420 due in 90 days is discounted at 6%.

Solution.

Face of note	= $420.
Bank discount 90 days	= $6.30.
Proceeds	= $420 — $6.30, or $413.70.

Example 2.—A note for $780 dated May 5, payable in 6 months, with interest at 6%, is discounted at 6%, August 3. Find the bank discount and proceeds.

Solution.

Face of note	$780.00
Interest for 6 months at 6%	23.40
Value of note at maturity	803.40
Bank discount for 94 days at 6% on $803.40	12.59
Proceeds = $803.40 — $12.59, or	$790.81

To find the face of a note, when the proceeds, time, and rate of discount are given.—To find the face of a note, when the proceeds, time, and rate of discount are given, divide the proceeds by the proceeds of $1 at the given rate and time.

Example.—For what sum must a non-interest bearing note be drawn, due in 90 days, so that when discounted at a bank at 6% per annum, the proceeds will be $537.40?

Solution.

$.015 = Bank discount on $1 for 90 days.
$1 — .015 = $.985, the proceeds of $1.
$537.40 ÷ $.985 = 545.58 times, or $545.58.

Problems.

1. Compute the bank discount and proceeds of each of the following notes:

(*a*) $1500.00 Jackson, Minn., Aug. 21, 1922.

Thirty days after date.....I.....promise to pay to the order of.....Rugg & Company..... Fifteen Hundred and.....No–100 Dollars.

Value received. Interest at 6%.

L. WAGNER

Discounted August 30, 1922, at 6%.

(*b*) $3,145.50 Minneapolis, Minn., July 15, 1922.

Three months after date.....I.....promise to pay to the order of.....Eli Perkins & Company..... Three Thousand One Hundred Forty-five and 50–100 Dollars.

Value Received. Interest at 6%.

E. L. MASON

Discounted July 15, 1922, at 8%.

(*c*) $1,480.00 Winona, Minn., May 20, 1922.

Six months after date.....I.....promise to pay to the order of.....J. H. Mosher..... Fourteen Hundred Eighty and.....No–100 Dollars.

Value received. Interest at 6%.

JAMES KING

Discounted July 15, 1922, at 6%.

(*d*) $135.00 St. Paul, Minn., June 13, 1922.

One year after date.....I.....promise to pay to the order of.....A. B. Cutts..... One Hundred Thirty-five and.....No–100 Dollars.

Value received. Without interest.

H. CARLSON

Discounted December 3, 1922, at 8%.

(*e*) $475.25 Moorhead, Minn., October 12, 1922.

Four months after date.....I.....promise to pay to the order of.....Otto A. Bade..... Four Hundred Seventy-five and.....25–100 Dollars.

Value received. Interest at 5%.

J. H. MOSHER

Discounted December 1, 1922, at 7%.

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2. A note dated May 13, 1922, payable in 4 months, without interest, was discounted May 31, 1922, at 8%. If the proceeds were $220.83, what was the face of the note?

3. A person owes $2,500. For what amount must he draw his note at 90 days without interest, so that when discounted at 6% the proceeds will cancel the debt?

Chapter 11

PARTIAL PAYMENTS

Part payments on debts.—A debtor who owes a large amount may by agreement make equal or unequal payments on the principal at regular or irregular intervals. The creditor should have interest on the loan, and it is fair that each payment made should draw interest in favor of the debtor.

Methods.—There are two methods of applying these payments of principal and interest to the reduction of the debt. The method adopted by the Supreme Court of the United States is termed the "United States Rule"; the other method, which business men have adopted by a custom more or less prevalent, is termed the "Merchants' Rule."

United States Rule.—The United States Rule, which is now a law in many of the States, having been made so either by statute or by court decision, has for its object a just settlement without allowing compound interest.

The court holds that when a part payment is made on an interest-bearing debt, the payment must first be used to discharge the accumulated interest, and what remains of the payment is to be applied in cancellation of the principal. But if the payment is smaller than the accumulated interest, no cancellation takes place, and the previous principal continues to draw interest until the accumulated payments exceed the accumulated interest.

Example.—An interest-bearing note for $1800 dated March 1, 1920, had the following indorsements:

September 27, 1920	$500
March 15, 1921	$ 25
June 1, 1921	$700

How much was due September 1, 1921?

Solution (U. S. Rule).

Face of Note March 1, 1920		$1,800.00
Interest on $1,800 at 6% from March 1, to Sept. 27, 6 months and 26 days		61.80
Amount due Sept. 27, 1920		$1,861.80
Deduct Payment		500.00
Balance due Sept. 27, 1920		$1,361.80
Interest on $1,361.80 at 6% from Sept. 27, to March 15, 5 months and 18 days, $38.13. As this interest is larger than the payment made at March 15, the interest is not added and the payment is not deducted.		
Interest on $1,361.80 at 6% from Sept. 27, to June 1, 1921, 8 months and 4 days		55.38
		$1,417.18
Deduct Sum of Payments: March 15	$ 25.00	
June 1	700.00	725.00
Balance due June 1, 1921		$ 692.18
Interest on $692.18 at 6% from June 1 to Sept. 1, 1921, 3 months		10.38
Balance due September 1, 1921		$ 702.56

Merchants' rule.—Find the amount of the debt (principal and interest) to the date of final settlement, or if for more than one year to the end of the first year. Deduct from this the sum of all the payments and interest on same to the date of settlement, or to the end of the year. The remainder will be the amount due at date of settlement, or at the beginning of the new year.

Example.—(For comparative purposes, the same problem will be used here as was used as an illustration of the United States Rule.)

Solution (Merchants' Rule).

Face of Note, March 1, 1920		$1,800.00
Interest, 1 year at 6% to March 1, 1921		108.00
		$1,908.00
Deduct:		
First payment, September 27, 1920	$500.00	
Interest at 6% to March 1, 1921, 5 months, 4 days	12.83	512.83
Balance due at Beginning of 2nd Year		$1,395.17
Interest on $1,395.17 at 6%, March 1 to September 1, 1921—6 months		41.86
		$1,437.03
Deduct:		
Second Payment, March 15, 1921	$ 25.00	
Interest at 6% from March 15 to Sept. 1, 1921, 5 months, 16 days	.69	
Third Payment, June 1, 1921	700.00	
Interest at 6% from June 1 to Sept. 1, 1921, 3 months	10.50	736.19
		$ 700.84

The difference of $1.72, between the balance as computed by the Merchants' Rule and the balance as computed by the United States Rule is small, but a much greater difference will occur when the time is long and the amount large.

It is usual to compute the balance due on notes of one year or less by the Merchants' Rule.

This method of computing partial payments compounds the interest at the end of the year, when the time of the note is for more than one year.

Problems.

The following problems should be solved first by the United States Rule and then by the Merchants' Rule, and the answers compared.

1. On a note of $3,500 with interest at 6%, dated October 3, 1920, were the following indorsements: January 1, 1921, $1,000; March 20, 1921, $390; June 2, 1921, $845. What was due September 16, 1921?

2. A note of $1,750, bearing 6% interest, dated August 27, 1919, for 9 months, was indorsed on November 10, 1919, $275; on February 13, 1920, $600; on May 3, 1920, $400. What was due at maturity?

3. A note for $1,600 at 6% interest is dated June 16, 1920, to run one year, and has the following indorsements: September 15, 1920, $250; March 10, 1921, $400; May 15, 1921, $600. What is due at maturity?

4. A note for $675, dated April 13, 1917, and bearing 6% interest, had the following indorsements: January 15, 1918, $150; June 12, 1918, $135; January 8, 1919, $65; October 13, 1919, $75; April 3, 1920, $45; September 1, 1920, $25. How much remained due January 1, 1921?

5. The following payments were made on a note of $3,150, dated August 12, 1916, drawing 6% interest and running five years from date: July 1, 1917, $100; September 5, 1918, $800; January 19, 1919, $156; April 1, 1921, $900; June 15, 1921, $750. How much was due August 12, 1921?

6. What was the amount due October 17, 1920, on a note for $1,000, dated May 2, 1919, and on which were the following indorsements, interest 7%: June 2, 1919, $80; December 15, 1919, $20; May 2, 1920, $32; June 2, 1920, $100?

7. How much was due January 2, 1921, on the following note? Payments: January 15, 1918, $160; October 12, 1918, $200; December 30, 1919, $180; April 3, 1920, $420; November 16, 1920, $300.

$2,400.00 Minneapolis, Minn., Jan. 15, 1917.

Two months after date.....I.....promise to pay to.....R. M. Jackson.....or order Two Thousand Four Hundred.....No–100 Dollars.

Value received, with interest at 6%.

W. A. SHURTLEFF

Chapter 12

AVERAGE

Simple average.—The simple average of a group of numbers is determined by adding the numbers to be averaged and dividing the sum by the number of numbers added.

Example.—Find the average rate per kilowatt hour for electrical energy from the following statistics:

New England States	2.88¢
South Atlantic States	2.77¢
Atlantic States	2.19¢
North Central States	1.88¢
Pacific Northwest	1.81¢

Solution.

$$2.88 + 2.77 + 2.19 + 1.88 + 1.81 = 11.53.$$
$$11.53 \div 5 = 2.306.$$

Explanation.—The number of items to be added is 5, and the sum is 11.53 cents. 11.53 cents divided by 5 equals 2.306 cents, the average rate per kilowatt hour.

Problems.

1. The week of January 14, 1924, the week's price range of July wheat at Minneapolis was:

Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
1.13⅝	1.13⅞	1.14⅝	1.14⅝	1.13¾	1.13½

What was the average price for the week?

2. January 20, 1924, Wells-Dickey Company reports bonds of particular interest to the Northwest as follows:

		Maturity	*Yield*
329,000	Dubuque, Iowa, School 4½'s	1929–38	4.50%
2,000,000	Des Moines Joint Stock Ld Bk 5's	1963	4.85%
180,000	Choteau, Montana, Refunding 6's	serial	5.40%
547,000	Grand Rapids, Mich., Street 5's	serial	4.50%
2,500,000	City of St. Paul 4½'s	1944–54	4.35%

If X invested in a $1000 bond in each of the above issues, what is the average yield on his investment?

3. The preliminary report on the cement industry released by the United States Geological Survey shows the following:

	1921	*1922*	*1923*
Production (in barrels).....	98,842,000	114,789,000	137,377,000
Shipments (in barrels).....	95,507,000	117,701,000	135,887,000
Stocks on hand December 31...	11,938,000	9,134,000	10,581,000

(*a*) What was the average yearly production for the three years?

(*b*) What was the average yearly shipment for the three years?

(*c*) What was the average stock on hand December 31, for the three years?

Moving averages.—Moving averages are a series of simple averages of statistics applicable to groups of an equal number of time units, each successive group excluding the first time unit of the preceding group and including the unit immediately following those of the preceding group. For instance, a yearly moving average, by months, might begin with an initial group including the twelve months of 1923. The next group would omit January, 1923, and include the remaining eleven months of 1923 and January, 1924.

Example.—The labor costs in a certain manufacturing plant for the first six months of 1923 were as follows:

January	$3,363.17
February	3,644.15
March	4,472.90
April	3,209.20
May	3,415.40
June	4,152.05

The labor costs for the next two months were:

July	$3,824.06
August	4,015.25

What has been the average labor cost for each six months since January 1, 1923?

Solution.

The labor cost for the period from January 1 to June 30 is the sum of the labor costs for each of the six months, or

$22,256.87. The average for the period is $22,256.87 ÷ 6, or $3,709.48.

The average for the period from February 1 to July 31 is computed as follows:

Total: January 1 to June 30.....	$22,256.87
Deduct: January labor cost.....	3,363.17
	18,893.70
Add: July labor cost.....	3,824.06
	$22,717.76

$22,717.76 ÷ 6 = $3,786.29.

The average for the period from March 1 to August 30 is calculated in the same manner.

Total: February 1 to July 31.....	$22,717.76
Deduct: February labor cost.....	3,644.15
	19,073.61
Add: August labor cost.....	4,015.25
	$23,088.86

$23,088.86 ÷ 6 = $3,848.14.

Comparison of these averages, $3,709.48, $3,786.29, and $3,848.14 shows an increase for each period.

For permanent records, these averages should be tabulated.

1923	*Labor Cost*	*Moving Average*	*Increase or Decrease* *
January-June	22,256.87	3,709.48	——
February-July	22,717.76	3,786.29	76.81
March-August	23,088.86	3,848.14	61.85

* Indicate decreases with asterisks.

Further comparisons, based on the figures of prior periods, may be made in succeeding years. A column may be annexed to show the increase or decrease of the average of each six months' period compared with the simple average for the previous year. Another column may be used to show the increase or decrease in the moving average for the current six months' period compared with the moving average for the same period of the previous year.

Problems.

1. If the labor costs of the company in the preceding example were as stated below for succeeding months, compute the moving averages.

September	$3,785.40
October	3,628.55
November	4,129.10
December	3,956.80

2. Using the averages obtained in Problem 1, complete the tabular record for the year 1923.

3. Find the simple average for the year 1923.

Progressive average.—The method of progressive average is cumulative. The results of the latest period are added to the total previously computed, and the amount is divided by the previous divisor plus 1.

Example.—Department *A* sales were: January, $5364; February, $4872; March, $5024. Department *B,* January, $2561; February, $2325; March, $2753. Find the progressive averages.

Solution.

SALES RECORD—1923

Dept.	*Jan.*	*Feb.*	*Total*	*Aver.*	*March*	*Total*	*Aver.*	*April*
A	5364	4872	10236	5118	5024	15260	5087	
B	2561	2325	4886	2443	2753	7639	2546	etc.

Explanation.—Department *A,* January and February sales total $10,236. $10,236 ÷ 2 = $5118, the average for the two months. $10,236 + $5024 = $15,260, total sales for the 3 months. $15,260 ÷ 3 = $5087, the average for the 3 months. The record for the year would be completed in this manner.

The totals and averages of Department *B* are computed in the same way.

Problem.

With the above record and the following information, complete the record for the six months' period.

Department *A* sales: April $6012; May, $5834; June, $5910.

Department *B* sales: April, $2631; May, $2234; June, $2596.

Periodic average.—Periodic average is simple average applied to statistics applicable to the same unit of time for several periods, and may be used to show a variation in expenses, earnings, sales, etc.

Illustration.

EXPENSES

Month	*1923*	*1922*	*1921*	*1920*	*Total*	*Average*
January	$478.60	$392.85	$429.65	$356.00	$1,657.10	$414.28
February	462.37	529.83	531.33	535.35	2,058.88	514.72
March	347.92	629.89	432.45	567.89	1,978.15	494.54

Explanation.—The expenses for January for the four years are totaled and the total, $1657.10, divided by 4, the number of years shown, equals $414.28, the average monthly expense for January. The other averages are calculated in the same manner.

Problem.

The payrolls of a company have been as follows:

Month	*1923*	*1922*	*1921*	*1920*	*Total*	*Average*
January	$5,675	$5,032	$4,957	$5,138		
February	5,413	5,261	3,920	4,076		
March	5,030	4,932	5,275	4,822		

Compute the periodic average.

Weighted average.—Weighted averages derive their name from the fact that they must be reduced to a common base in order to produce the correct average.

The average-price method of pricing requisitions in cost accounting is illustrated in the following example:

Example.—A stock record shows the following receipts:

4800 pounds at 20¢
3000 pounds at 18¢
4000 pounds at 21¢

What is the average price at which to cost requisitions for the month?

Solution.

4800	pounds	at 20¢ =	$ 960
3000	pounds	at 18¢ =	540
4000	pounds	at 21¢ =	840
11800	pounds		$2340

$2340 ÷ 11800, the number of pounds, = 19.83 cents per pound average price.

Example.—A manufactured product is composed of four ingredients in the following ratios and costs per pound:

Material	*Pounds*	*Price per Pound*
#1	1	$1.50
#2	3	.75
#3	4	1.25
#4	2	2.00

Owing to price fluctuations, it was found in the second year that the raw material costs had increased as follows:

Material	*Per Cent*
#1	50%
#2	100%
#3	10%
#4	25%

What was the average per cent of increase in the cost of raw material composing the finished product?

Solution.

Material	*Pounds*	*Cost per Lb.*	*Total Cost*	*Per cent Price Increase*	*Increased Cost*
#1	1	$1.50	$ 1.50	50	$ .75
#2	3	.75	2.25	100	2.25
#3	4	1.25	5.00	10	.50
#4	2	2.00	4.00	25	1.00
			$12.75		$4.50

4.50 ÷ 12.75 = 35.29%, weighted average per cent.

Problems.

1. *B* owns the following securities:

$1000	Alabama Power Company	6's
500	Pacific Gas & Electric Co.	5½'s
2000	St. Paul Union Depot Co.	5's
1000	Mooresville, N. C., Water	5¼'s
2000	Clay County, Minn.	4¾'s

What is the average rate of interest earned on his investment assuming the securities were bought at par?

2. Calculate the average price of the following for the purpose of pricing requisitions:

2500 pounds at 35¢
1000 pounds at 33¢
2000 pounds at 31¢
500 pounds at 36¢

3. "Salto" is manufactured from the following materials, used in the ratios given:

Material	*Pounds*	*Price*
A	5	$ .25
B	4	.30
C	6	.20
D	1	1.50

If the raw materials advance in price at the following rates, what will be the average per cent increase in the cost of the finished product?

Material	*Per cent Increase in Price*
A	100%
B	50%
C	75%
D	20%

Chapter 13

AVERAGING DATES OF INVOICES

Definition.—Averaging dates of invoices is the process of finding the date when several invoices due at different dates may be paid in one amount without loss to either debtor or creditor. This date is called the equated date.

Use.—Averaging the dates of invoices is most frequently encountered in bankruptcy settlements where claims when filed with a trustee must show the average due date of the items if interest is to be secured on overdue amounts. In general, the equated date is important in the settlement of bills of long standing and in fixing the date of a note in settlement of invoices.

Term of credit.—A term of credit is the time elapsing between the date of the bill and the date on which it becomes due; as, "Bill purchased January 10, Term of Credit 10 days." The due date would be January 20, the date from which interest calculations are made.

Average due date.—The average due date is the date on which settlement of the complete account should be made by paying the amount of the invoices without charge for interest on overdue items or allowance for discount on prepaid items.

Focal date.—The focal date is an assumed date of settlement with which the due dates of the several items may be compared, in order to determine the equated date.

Any date may be used as the focal date, and the final result will be the same. In the interest method any rate per cent may be used, and the result will be the same. Six per cent is usually used, as the computations are less complicated.

In all calculations use the nearest dollar. For instance, in $115.29, use $115, and in $161.84, use $162.

When several bills are sold, some having a term of credit, first find the due date of those with a term of credit, and then find the equated date of the several bills.

Do not use a fraction of a day in determining the average date.

Methods.—There are two methods in common use: the Product Method and the Interest Method.

Rule for product method.—Assume the earliest due date as the day of settlement for the items. Multiply each item by the number of days intervening between the assumed date and the due date of the item, and divide the sum of the several products by the sum of the account. Count forward from the assumed date the number of days obtained in the quotient. The result will be the average due date.

Example.—At what date may the following bills of merchandise be paid in one amount without loss to either party? Due January 1, $150; February 14, $200; April 20, $155; June 15, $200.

Solution by Product Method.

Due January 1	$150	×	0	= 0
Due February 14	200	×	44	= 8800
Due April 20	155	×	109	= 16895
Due June 15	200	×	165	= 33000
	705			58695

58695 ÷ 705 = 83 days.

83 days after January 1 is March 25.

Explanation.—For convenience, take the earliest due date as the time of settlement. Should the first bill, which is due January 1, be paid on that date, there would be neither gain nor loss to either party. On the second bill there would be interest from January 1 to February 14, or 44 days, which is equivalent to interest on $8800 for one day. On the third bill there would be interest from January 1 to April 20, or 109 days, which is equivalent to interest on $16,895 for one day. On the fourth bill there would be interest from January 1 to June 15, or 165 days, which is equivalent to interest on $33,000 for one day.

If all the bills were paid January 1, there would be interest on $58,695 for one day, or $705 for 83 days. It is evident that the bills could be paid at a time 83 days later than January 1, or March 25, without loss to either party.

Verification.

The interest on $150 for 83 days is	$2.08
The interest on $200 for 39 days is	1.30
Total gain of interest to debtor	$3.38
The interest on $150 for 26 days is	$ .67
The interest on $200 for 82 days is	2.72
Total gain of interest to creditor	$3.39

The gain of interest to the debtor is on all bills paid after they are due.

The gain of interest to the creditor is on all bills paid before they are due. These two results should be equal, or within a few cents of the same amount. The reason for a little discrepancy is in the fraction of a day in determining the due date of the account.

Solution by Interest Method.

January 1	$150 for	0 days	= $ .00	interest
February 14	200 for	44 days	= 1.47	"
April 20	155 for	109 days	= 2.82	"
June 15	200 for	165 days	= 5.50	"
Total interest			= $9.79	

Interest on $705 for 1 day is $.1175.

$9.79 ÷ $.1175 = 83 times, or 83 days.

Explanation.—Assume January 1, the earliest due date, to be the time of settlement. If the amount of the account, $705, is paid January 1, there will be a loss of interest to the debtor of $9.79. The interest on $705 for one day at 6% is $.1175. It will take a principal of $705 as many days to produce $9.79 as $.1175 is contained in $9.79, or 83 days, the same result as was obtained by the product method.

Problems.

1. At what date may the following bills be paid in one amount without loss to either party? Due May 16, $215; June 24, $160; August 11, $90; August 25, $950.

2. What is the average due date of the following bills? Due, April 10, $350; May 13, $500; June 28, $450; July 15, $55; August 13, $400.

3. What is the equated date for the payment of the following bills? Due, March 15, $135; April 20, $690; May 11, $385; June 29, $650; July 23, $550; August 20, $340.

4. Sold a customer the following bills of goods on an average credit of 60 days each: June 1, $165.25; June 18, $181; July 13, $252.15; July 28, $95.65; August 14, $312.50; August 29, $781.70. When may the whole indebtedness be equitably discharged in one payment?

5. A commission merchant sold several bills of goods, on a credit of 3 months, as follows: April 18, $376; May 13, $243.90; May 20, $634.25; June 20, $264.35; July 23, $812.90; August 20, $284.75. What was the equated date of payment?

6. Obtain the average date of maturity of the following sales:

Oct.	1	$ 261.14 on 30 days credit.
Oct.	8	394.85 on 60 days credit.
Oct.	22	821.66 on 90 days credit.
Nov.	27	915.60 on 90 days credit.
Nov.	13	1,000.00 on 60 days credit.
Dec.	16	1,580.00 on 6 mos. credit.

Chapter 14

EQUATION OF ACCOUNTS OR COMPOUND AVERAGE

Definition.—Equation of accounts or compound average is the process of finding the date when the balance of an account having both debits and credits can be paid without loss to either debtor or creditor.

Rule for the product method.—After finding the due date of each entry, take the earliest due date as the focal date for both sides of the account. Find the number of days between the focal date and the due date of each entry; obtain the product of each number of days multiplied by the principal to which it applies. Find the sum of the products on both the debit and credit sides of the account. Divide the difference between the sums of the debit and the credit products by the balance of the account. The quotient will be the number of days between the focal date and the average date of settlement.

When to date forward or backward.—The average date is forward from the focal date when the balance of the account and the excess of the products are on the same side (both debits or both credits); if on the opposite side, the average date is backward from the focal date.

Example.—At what date may the balance of the following account be paid without loss of interest to either party?

Dr.			*Cr.*		
1922			1922		
Jul. 1	Mdse. 30 days	$250	Aug. 15	Cash	$400
Jul. 26	Mdse. 30 days	425	Sep. 10	Cash	300
Aug. 15	Mdse. 60 days	320	Sep. 20	Cash	150
Aug. 30	Mdse. 60 days	500			

Solution by Product Method.

Due	*Dr.*		*Cr.*	
Jul. 31,	$ 250 × 0 =	00000		
Aug. 25,	425 × 25 =	10625	Aug. 15, $400 × 15 =	6000
Oct. 14,	320 × 75 =	24000	Sep. 10, 300 × 41 =	12300
Oct. 29,	500 × 90 =	45000	Sep. 20, 150 × 51 =	7650
	$1495	79625	$850	25950

Dr. side	$1,495	$79,625
Cr. side	850	25,950
	$ 645	$53,675

$53,675 ÷ $645 = 83 days after July 31, or October 22.

Explanation.—First find the due date of each item. For convenience, assume July 31, the earliest due date, as the day of settlement for all the items on each side of the account. Proceed as in averaging dates, the process of which was described in the previous chapter. Using the earliest date, July 31, for the debit side, there is the equivalent of a loss of interest on $79,625 for 1 day, and a gain on the credit side equivalent to the interest on $25,950 for 1 day; or a net loss of interest on $53,675 for 1 day, or on $645 for 83 days. It is evident that the date when there would be no loss to either party must be 83 days after July 31, or October 22.

Solution by Interest Method.

Due Date		*Dr.*
Jul. 31,	0 days interest on $ 250 =	$ .00
Aug. 31,	25 days interest on 425 =	1.77
Oct. 14,	75 days interest on 320 =	4.00
Oct. 29,	90 days interest on 500 =	7.50
	$1495	$13.27

Date		*Cr.*
Aug. 15,	15 days interest on $400 =	$1.00
Sep. 10,	41 days interest on 300 =	2.05
Sep. 20,	51 days interest on 150 =	1.28
	$850	$4.33

Dr.	$1495	Int.	$13.27
Cr.	850	Int.	4.33
6)	645		$8.94
	$.1075	Interest for 1 day	

$8.94 ÷ .1075 = 83 days after July 31.

83 days after July 31 is October 22, the equated date of payment.

The explanation of the product method is applicable to the interest method. The variance is in finding interest on each item and dividing the interest balance by the interest for one day on the balance of the account.

Problems.

Find the equated date:

1.

E. B. SMITH

Dr.			Cr.		
Aug. 1,	Mdse.	$800	Aug. 20,	Cash	$300
Sep. 20,	Mdse.	400	Sep. 30,	Cash	500
Oct. 30,	Mdse.	900	Nov. 24,	Cash	700

2.

H. M. EVERETS

Dr.			Cr.		
Apr. 10,	Mdse. 30 days	$480	May 24,	Cash	$300
May 4,	Mdse. 60 days	560	Jun. 20,	Cash	200
Jul. 15,	Mdse. 60 days	780	Aug. 4,	Cash	700
Oct. 22,	Mdse. 60 days	920	Dec. 1,	Cash	800

3.

A. H. BAKER

Dr.			Cr.		
Jan. 3,	Mdse. 3 mo.	$600	Feb. 20,	Cash	$400
Feb. 20,	Mdse. 20 days	250	Mar. 3,	Note 10 days	300
Mar. 15,	Mdse. 60 days	380	Apr. 16,	Note 10 days	800
Mar. 24,	Mdse. 4 mo.	420	Apr. 24,	Note (with Int.) 60 days	150
Apr. 11,	Mdse. 1 mo.	800			

4.

Minneapolis, Minn., November 16, 1922.

BUELL & MAETZOLD

Produce and Commission Merchants

Account Sales of Merchandise.

Sold on Account of Bishop & Brooks.

Aug. 1,	100 boxes	Lemons	at $ 2.95
Sep. 20,	240 gallons	Molasses	at .37
Sep. 30,	120 boxes	Oranges	at 3.15
Nov. 12,	200 barrels	Pork	at 14.30

Charges

Aug. 1,	Advances	$2,000.00
Aug. 2,	Insurance	1.61
Aug. 3,	Storage	12.87
Aug. 4,	Freight and Cartage	96.00
Nov. 12,	Commission	5%

Chapter 15

ACCOUNT CURRENT

Definition.—An account current is a transcript of the ledger account together with the details taken from the books of original entry. It shows the dates on which sales were made, the term of credit of each item, cash payments and, if settled by note, the date of the note and other details.

Methods.—Two methods are used for finding the amount due: the Interest Method and the Product Method.

Example.—Find the balance due January 1, 1922, with interest at 6%, on the following ledger account.

J. B. JOHNSON

	Dr.			Cr.	
1921			1921		
Sep. 1,	Bal.	$1,200	Oct. 1,	Cash	$1,000
Sep. 20,	Mdse. 30 days....	400	Nov. 10,	Cash	200
Oct. 30,	Mdse. 30 days....	520	Dec. 3,	Cash	400
Nov. 25,	Mdse. 30 days....	350	Dec. 15,	Note 10 days.....	300

Solution by Interest Method.

J. B. JOHNSON

	Dr.			Cr.		
Date Due	*Amount*	*Days*	*Interest*	*Date*	*Amount*	*Days*	*Interest*
Sep. 1	$1,200	122	$24.40	Oct. 1	$1,000	92	$15.33
Oct. 20	400	72	4.87	Nov. 10	200	52	1.73
Nov. 29	520	33	2.86	Dec. 3	400	29	1.93
Dec. 25	350	7	.41	Dec. 25	300	7	.35
	$2,470		$32.54		$1,900		$19.34
	1,900		19.34				
	$ 570	+	$13.20	= $583.20.			

Explanation.—The number of days opposite each entry is the actual number of days from the date of the item to January 1, 1922, the date as of which the balance of the account is desired.

Solution by Product Method.

Dr.					Cr.			
Date Due	*Amount*	*Days*	*Product*		*Date*	*Amount*	*Days*	*Product*
Sep. 1	$1,200	122	$146,400		Oct. 1	$1,000	92	$ 92,000
Oct. 20	400	73	29,200		Nov. 10	200	52	10,400
Nov. 29	520	33	17,161		Dec. 3	400	29	11,600
Dec. 25	350	7	2,450		Dec. 25	300	7	2,100
	$2,470		$195,210			$1,900		$116,100
	1,900		116,100					
	$ 570		$ 79,110					

The interest on $79,110 for 1 day = $ 13.20.
$570 + $13.20 = $583.20.

In some instances it is more convenient to find the equated due date and calculate the interest on the balance of the account, from that date to the date of settlement.

Problems.

1. Find the equated date for the balance of the following account, and the amount that will be required to settle the account on the first of May, 1910, interest being charged at the rate of 6% per annum:

Dr.				Cr.			
1910				1910			
Jan. 1,	Mdse. at 30 days...	$800		Feb. 1,	Cash.....	$500	
Jan. 20,	Mdse. at 30 days...	500		Mar. 15,	Cash.....	275	
Feb. 1,	Mdse. at 30 days...	700					
Feb. 20,	Mdse. at 30 days...	600					

C. P. A.

2. What is the equated date for the payment of the balance of the following account (30 days to the month and 6% per annum)?

HENRY M. DOREMUS

Dr.				Cr.			
1904				1904			
Mar. 16,	Mdse. 4 mos....	$444.57		Jul. 1,	Cash	$400.00	
Mar. 30,	Mdse. 60 days...	376.82		Jul. 20,	Cash	375.00	
Apr. 20,	Mdse. 30 days...	712.19		Aug. 16,	Cash	700.00	
May 17,	Mdse. 4 mos....	628.75		Aug. 30,	Cash	600.00	
May 28,	Mdse. 4 mos....	419.31					

Henry M. Doremus desires to settle the above account on September 13, 1904. What amount of money shall he pay?—*C. P. A.—New Jersey.*

3. In auditing an account, the auditor finds that Robert Brown had bought a bill of goods amounting to $500, payable on August 10, less 2%. He had, however, made payments thereon as follows:

Jun. 2	$100
Jun. 15	100
Jul. 3	100

On what date would he be required to make payment of the remaining $200 to entitle him to the 2% discount under the original terms of sale?—*C. P. A.—New York.*

4. What is the average date of the following account?

JAMES HENDERSON

1908			1908		
Mar. 9,	To Mdse.....	$ 300.00	Mar. 20,	By Cash.....	$247.00
May 12,	To Mdse.....	474.00	May 14,	By Cash.....	400.00
Jun. 19,	To Mdse.....	564.00	Jul. 10,	By Cash.....	260.00
		$1,338.00			$907.00

—C. P. A.—Maryland.

5. What is the average date of each of the following?

(*a*) $114 due April 10; $140 due April 26; $320 due May 22; $976 due June 6.

(*b*) June 3, $375 on 30 days' time; June 28, $420 on 60 days' time; July 16, $560 on four months' time; September 4, $228 on 90 days' time.

(*c*) Dr. May 16, $437. Cr. May 23, $400.
Dr. May 31, $324. Cr. June 16, $300.

C. P. A.—Michigan.

Chapter 16

STORAGE

Definition.—Storage is the charge made by a warehouse or depositary for the keeping of goods until required for use or for transportation to some other point.

Running account.—When goods are being received and delivered for a company, the storage company keeps a running account showing the dates at which goods are received and delivered, together with the number of packages, barrels, etc. Storage is charged for the average number of days for which one package, barrel, or box has remained in storage. The average number of days is divided by 30 to reduce the average number of days to months, or by 7 to reduce the average number of days to weeks, as the case may be; then the number of months or weeks is multiplied by the price per term.

Example.—The following is a memorandum of salt stored by H. F. Adams with the Twin City Storage Company at 4¢ per barrel per term of 30 days average storage.

Date	*Receipts*	*Deliveries*	*Balance*	*Time in Storage*	*Equivalent for 1 Day*
Jun. 4	120 bbl.		120 bbl.	28 days	3360 bbl.
Jul. 2		20 bbl.	100 bbl.	18 days	1800 bbl.
Jul. 20	100 bbl.		200 bbl.	10 days	2000 bbl.
Jul. 30		50 bbl.	150 bbl.	11 days	1650 bbl.
Aug. 10		100 bbl.	50 bbl.	15 days	750 bbl.
Aug. 25		50 bbl.	0 bbl.		
					9560 bbl.

9560 barrels for 1 day are equivalent to 1 barrel for 9560 days, and 9560 divided by 30 (the number of days per term) equals 318⅔ terms. In some cities a full month's storage is charged for any part of a month that goods remain in store; in others, 15 days or less are called one-half of a month, and over 15 days a whole month. 318⅔ terms would be charged

for as 319 terms, and $319 \times .04 = 12.76$. Therefore, \$12.76 is the storage charge.

Problems.

1. Compute storage at 4¢ per barrel per term of 30 days average storage on the following memoranda:

Received		*Delivered*	
Feb. 16	200 barrels	Feb. 22	100 barrels
Feb. 21	250 barrels	Mar. 9	250 barrels
Mar. 18	400 barrels	Mar. 21	300 barrels
Mar. 26	200 barrels	Apr. 14	400 barrels

2. A potato grower stored 4000 bushels of potatoes with the Red River Valley Storage Company at 5½¢ per cwt. per month. The following is a memorandum of the transactions that occurred. Compute the amount of storage.

Received		*Delivered*	
Sep. 20, 1922	1500 bushels	Dec. 4, 1922	500 bushels
Oct. 10, "	1000 bushels	Jan. 10, 1923	600 bushels
Oct. 15, "	1500 bushels	Jan. 24, "	500 bushels
		Feb. 20, "	1000 bushels
		Mar. 8, "	500 bushels
		Apr. 2, "	900 bushels

Chapter 17

FUNDAMENTALS OF ALGEBRA

Explanation.—The work of an accountant is so complicated in many details and requires so many technical calculations, that the numerous difficulties are almost impossible to overcome without a good foundation in mathematics. It is extremely difficult to solve many mathematical calculations by arithmetic. On the other hand, if the fundamentals of algebra are understood, the calculations may be made with comparative ease. Only the common and more useful principles of algebra are included in this work.

Symbols and terms.—In algebra the letters of the alphabet are often employed as symbols to represent numbers.

The following signs have the same meaning in algebra as in arithmetic:

$+$ is read "plus."
$-$ " " "minus."
$\times$ " " "times" or "multiplied by."
$\div$ " " "divided by."
$=$ " " "equals."

An exponent is a letter or number written at the right of another number and a little above to show how many times the number is to be used as a factor; and indicates the power of the number. Example, "25^3." When no exponent is indicated, 1 is understood to be the exponent.

The parenthesis ( ), the bracket [ ], and the brace { } indicate that the expression is to be taken as a unit.

An equation is an expression of equality between two numbers.

An axiom is a self-evident truth. Algebraic operations make use of the following axioms:

1. The value of an equation is not changed by adding the same number to both sides.

2. The value of an equation is not changed by subtracting the same number from both sides.

3. The value of an equation is not changed by multiplying both sides of an equation by the same number.

4. The value of an equation is not changed by dividing both sides of the equation by the same number.

Positive and negative numbers.—A positive or negative state of any concrete magnitude may be expressed without reference to the unit; thus, numbers that are greater than zero are positive and numbers that are less than nothing, or zero, are negative.

A unit which has a "+", or positive, sign prefixed to it is called a positive number; thus, + 5. If a "—" sign precedes the unit, it is called a negative number, and is written thus, — 5.

Addition of positive and negative numbers.—The result of addition is called a "sum" regardless of whether it is a positive or negative quantity.

Illustration.

The sum of + 6 and — 4 is + 2.
" " " + 4 *a* and — 2 *a* is + 2 *a*.
" " " + 2 and — 6 is — 4.
" " " + 3 *a* and — 7 *a* is — 4 *a*.
" " " — 4 and — 3 is — 7.
" " " — 3 *a* and — 4 *a* is — 7 *a*.

From the above, the following rules may be derived:

1. To add two numbers or terms of different signs, subtract the smaller number or term from the larger and prefix the sign of the larger.

2. To add two negative numbers add their absolute values and prefix the negative sign.

Example.

Add + 4 *a*, — 3 *a*, + 6 *a*. The problem may be restated thus: $(+4a) + (-3a) + (+6a)$.

Solution.

$$\begin{array}{r} +4a \\ +6a \\ \hline +10a \\ -3a \\ \hline +7a \end{array}$$

The addition of positive quantities is made in a manner similar to that of ordinary addition.

The addition of a positive and a negative number is equivalent to deducting the smaller number from the larger, and retaining the sign of the larger.

Problems.

Find the sum of the following:

1. $+6, -3, -4, +5, +3, -10.$ $(=-3)$
2. $-4, -6, -3, +5, +3, -4.$
3. $+1, -6, +3, -4, -6, -5.$
4. $a, +2a, +4a, +3a, -3a.$ $(=+7a)$
5. $3a, -13a, +4a, -6a, -5a.$
6. $3x, -4x, +5x, -3x, -6x.$
7. $3xy, -4xy, +5xy, +6xy.$

In adding terms which are multiples of the same letter, add the coefficients * of these terms, and unite the others with their proper signs. Thus $+6a, +7b, -5a, +6b$ becomes:

	$+6a$	$+7b$
	$-5a$	$+6b$
Added	a	$+13b$

Problems.

Find the sum of:

1. $+4a, -4b, +4a, +3a, -6b.$
2. $-2b, -2c, +4b, +6c.$
3. $9x, -3c, -3c, +3d.$
4. $5c, +4e, +3a, -6c, -4e.$
5. $6a, -7b, -4c, +3a, +7b.$

Note.—Hereafter when a term is not preceded by a positive or a negative sign, it is understood to be positive.

Parentheses, brackets, or braces.—Parentheses, brackets, or braces are used to indicate that the part inclosed is to be used as a single term or a single unit.

As the same rules apply to all signs of aggregation, in the explanation given hereafter, only parentheses will be mentioned.

* Coefficient—"a number or letter put before a mathematical quantity, known or unknown, to show how often it is to be taken."

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Rules.—(*a*) When a term in parentheses is preceded by a "+", or positive, sign, the parentheses may be removed without changing the signs of any of the inclosed terms.

(*b*) If a term in parentheses is preceded by a "—", or negative sign, on the removal of the parentheses it is necessary to change each of the positive and negative signs of the terms inclosed.

(*c*) If a term or a number of terms are to be inclosed in parentheses which are preceded by a positive sign, there need not be any change in the signs of the terms inclosed.

(*d*) If a term or number of terms are inclosed in parentheses which are preceded by a negative sign, it is necessary to change the sign of each of the terms inclosed.

Illustrations.—The expression should be written:

Before removing parentheses:	*After removing parentheses:*
$a+b+(c-d)$	$=a+b+c-d.$
$a+b-(c-d)$	$=a+b-c+d.$
$(a-b)+(c-d)-(e-f)$	$=a-b+c-d-e+f.$
$(4a+3b-c)-(d+4e-f)$	$=4a+3b-c-d-4e+f.$

Wherever possible, like terms should be combined; as,

$$(3a-3b+4c)-(2a-3c)=a-3b+7c.$$

If several algebraic expressions are inclosed one within the other by inclosure signs, such as parentheses or brackets, eliminate the innermost pair first.

First Illustration.

$$\begin{aligned} &a+[b-(c-d)]= \\ &a+[b-c+d] \quad = \\ &a+b-c+d. \end{aligned}$$

Second Illustration.

$$\begin{aligned} &a+b-[-(b+c)+(c-d)]= \\ &a+b-[-b-c+c-d] \quad = \\ &a+b+b+c-c+d \quad = \\ &\text{Combining, } a+2b+d. \end{aligned}$$

Problems.

Simplify the following by removing all signs of aggregation.

1. $3a + (4b - c) - (6a + b + c)$.
2. $3x + (6x - y) - (x - y - z)$.
3. $21 - [45 + (6 + 4) - 8]$.
4. $(8a - 4b) + (6c + d) - (3a + b + c)$.
5. $3a + [8y + (x - 4a) + 6y] - x$.

Subtraction.—Subtraction is the process of determining one of two numbers when their sum and one of the numbers are given.

The Minuend is the sum of two numbers.
The Subtrahend is the number to be deducted.
The Remainder is the required number.

To subtract, change the sign of the subtrahend and add the result to the minuend.

Example 1.—Subtract $8x$ from $4x$.

Solution.

$$(+4x) - (+8x) =$$
Removing parentheses, $+4x - 8x \quad = -4x.$

Example 2.—Deduct $-8x$ from $-4x$.

Solution.

$$-4x - (-8x) =$$
Removing parentheses, $-4x + 8x \quad = +4x.$

Example 3.—Deduct $-8x$ from $4x$.

Solution.

$$4x - (-8x) =$$
Removing parentheses, $4x + 8x \quad = +12x.$

To subtract one algebraic expression having two or more terms from another, change the sign of each term of the subtrahend and proceed as in addition.

Example.—Subtract $3a - 5c - 3d$ from $7c - 16a - 2d$.

Solution.

The changing of the signs of the subtrahend is usually done *mentally;* thus,

$$\begin{array}{r} -16a + 7c - 2d \\ 3a - 5c - 3d \\ \hline -19a + 12c + d \end{array}$$

Problems.

Subtract:

1. $14x$ from $3x$.
2. $14x$ from $-16x$.
3. $31a$ from $14a - 13b$.
4. $-7x$ from $6x$.
5. $-8a + 8b$ from $16a + 8b$.
6. $a - 3b + 4c - d$ from $-4a + 14b - 2c + 4d$.
7. $13a + 14c - 2b$ from $20b - 4c - 14a$.
8. $18a + 14b - 3c$ from $8a - 3b + 7c$.
9. $2x - 3y + 4z$ from $5x - 3z$.
10. $2p + 3q$ from $4p - 4q$.

Multiplication.—When two or more numbers are multiplied, the result is called a product.

When two numbers with *like* signs, either positive or negative, are multiplied, the product will be positive.

Illustrations.

$$(+a) \times (+b) = +ab.$$
$$(-a) \times (-b) = +ab.$$
$$(-4) \times (-4) = +16.$$

When two numbers with *unlike* signs, are multiplied, the product will be negative.

Illustrations.

$$(+a) \times (-b) = -ab.$$
$$(+4) \times (-3) = -12.$$

The exponent to be used in the product is equal to the sum of the exponents used in the multiplier and the multiplicand.

Illustrations.

$$(a)^2 \times (a)^1 = a^3.$$
$$(+a^4) \times (-a^3) = -a^7.$$
$$(-3a^2) \times (-2a^3) = +6a^5.$$

Problems.

Multiply the following:

1. $7a^2$ by $2a^5$.
2. $-4a^2$ by $-7a^3$.
3. $3x$ by $-9x$.
4. $3x + 4y - z$ by 4.
5. $2a - 4b + 8c$ by 12.
6. $3000 - s - b$ by 10.

7. $(4000 - x) - b$ by 12.
8. $B - (2000 - x)$ by 6.
9. $[20000 - (2000 + x)]$ by 10.
10. $[15000 - (2000 - T - B)]$ by $12\frac{1}{2}$.
11. $[20000 - (2000 - T - B) - B]$ by 10 y.
12. $[30000 - x - (2000 - B - x)]$ by 6 xy.

Division.—Division is the process of finding one of two numbers when the other number and their product are given. When the dividend and divisor have *like* signs, either positive or negative, the quotient will be a positive number.

Illustrations.

$$ab \div a = b.$$
$$-ab \div -b = a.$$
$$-14 \div -7 = 2.$$

When the dividend and the divisor have unlike signs, the quotient will be a negative number.

Illustrations.

$$-ab \div a = -b.$$
$$ab \div -b = -a.$$
$$-10 \div 5 = -2.$$
$$10 \div 5 = 2.$$

The exponent to be used in the quotient is equal to the difference between the exponent in the dividend and the exponent in the divisor.

Illustrations.

$$a^7 \div a^3 = a^5.$$
$$-ab^2 \div b = -ab.$$
$$8a^5b^4 \div -2ab^3 = -4a^5b.$$
$$20^5 \div 20^3 = 20^2.$$

Problems.

Divide the following:

1. 75 by 5.
2. -90 by 15.
3. -64 by -4.
4. 64 a by 4 a.
5. 40 a^2b by $-5a$.
6. $5a^2 - 60b$ by 5.
7. $6000 - (4000 - 5x)$ by 5.
8. $60a^{24} - 30a^{22} + 15a^{10}$ by $15a^5$.
9. $40000 - (10000 - 5x - 5y)$ by 5.
10. $72x^8 - 9x^{10} + 63x^6$ by $9x^4$.

Chapter 18

EQUATIONS

Simple equations.—"An equation is an expression of equality between two magnitudes or operations."—Webster. The members of the equations are separated by the sign of equality, "=", which means "is equal to." Either member of an equation may contain numerals, letters, or both. A simple equation is an equation of the first degree, and may contain but one unknown.

Illustration.

Simple equation, $x = 50.$
Or, $10x = 500.$

If the same number is added to or subtracted from both sides of an equation, the equality will not be destroyed.

Illustrations.

Simple equation, $x = 50.$
Adding 10 to each side, $x + 10 = 60.$

Simple, equation, $10x = 500.$
Subtracting 10 from each side, $10x - 10 = 490.$

If both sides of an equation are multiplied by or divided by the same number, the equality will not be destroyed.

Illustration 1.

$$3x = 15.$$
Multiplied by $$\underline{\qquad 5}$$
$$15x = 75.$$

Simplifying, or dividing by 15, $x = 5.$

Illustration 2.

$$4)\ 20x = 40$$
Divided by 4, $$5x = 10$$

Simplifying, or dividing by 5, $x = 2.$

Any term of either member of the equation may be transposed from one side to the other by changing the sign of the

term, and the value of the equation will not be affected. This operation is equivalent to either adding or subtracting the same quantity from both sides of the equation.

Illustration 1.

Simple equation,	$10x - 10 = 490.$	
Transposing the "— 10" to the right side of the equation, and changing the sign,	$10x$	$= 490 + 10.$
or:	$10x$	$= 500.$

Illustration 2.

Simple equation,	$x + 5 = 15.$	
Transposing,	x	$= 15 - 5.$
Or,	x	$= 10.$

Example.

Solve the equation,	$10x - 14$	$= 6x + 2.$
Transposing "+ $6x$" to the left side and changing the sign to a "—,"	$10x - 6x - 14 = +2.$	
Transposing "— 14" to the right side and changing the sign,	$10x - 6x$	$= +14 + 2.$
Uniting similar terms,	$4x$	$= 16.$
Dividing by 4,	x	$= 4.$

Verification.

Left side of equation:			*Right side of equation:*	
	$10x =$	40	$6x =$	24
	$-14 = -$	14	$+2 = +$	2
Adding,		26	Adding,	26

Therefore the two sides of the equation are equal, and $x = 4$.

Problems.

In the following equations solve for the unknown quantities and check the results:

1. $20a - 10 = 5 + 5.$
2. $5a + 5 = 55.$
3. $4b + 2 = 3b + 9.$
4. $32 + 17a = 22 + 27a.$
5. $9c - 23 = 19c - 18.$
6. $13 - 6x = 13x - 6.$
7. $24x - 46x + 110 = 30x - 150.$
8. $8a - 5(4a + 3) = 1 - 4(2a - 7).$
9. $-3 - 4(2b - 7) = 8b - 5(4b + 3).$
10. $18x - (10x - 8) = 20x - (6x + 4).$

Example.—How can 90 be divided into two parts so that one part shall be four times the other?

Solution.

Let	x = The smaller part.
Then	$4x$ = The larger part.
Adding,	$5x = 90$.

Dividing each member of the equation by 5, $x = 18$, the smaller part.
$4x = 72$, the larger part.

Example.—How many dimes and cents are there in $2.40 if there are 60 coins in all?

Solution.

Let	x = The number of dimes.
Then	$60 - x$ = The number of cents.
Therefore	$10x$ = The value of the dimes,
But	$60 - x$ = The value of the cents.
Simplifying,	$10x + 60 - x = 240$.
Transposing,	$9x = 180$.
	$x = 20$, or there are 20 dimes.
	$60 - x = 40$, or there are 40 cents.

Verification.

20 dimes	=	$2.00
40 cents	=	.40
60 coins	=	$2.40

Example.—The *P Q* Company wrote off depreciation on their building which cost $50,000, at the rate of 2% of the original cost per annum. This amount was included in General and Administration Expense account and constituted $\frac{1}{10}$ of this expense.

The purchases cost two and one-half times the old inventory. The value of the old inventory was two times the amount of the Selling Expense; the Selling and General Administration Expense account each equalled 10% of the Sales. The new inventory was valued at an amount equal to the Selling Expense.

The Interest and Discount costs were $\frac{1}{5}$ of the Selling Expenses. Set up a Profit and Loss Statement showing the net profit from operation.

Solution.

	$50,000 × 2%	= $1,000, depreciation.
	$1,000, depreciation	= 1/10 of genl. and adm. expense.
Multiplying by 10,	$1,000 × 10 = $10,000	= General and adm. expense.
	$10,000	= Selling Expenses.
	Selling Expenses	= 1/10 of Sales.
Multiplying by 10,	$10,000 × 10	= $100,000, or Sales.
	2 × $10,000	= $20,000, or old inventory.
	2½ × $20,000	= $50,000, purchases.
	1/5 of $10,000	= $2,000, interest and discount.

THE *P Q* COMPANY

PROFIT AND LOSS STATEMENT

December 31, 1922

Sales		$100,000
Costs:		
Old Inventory	$20,000	
Purchases	50,000	
Total	70,000	
Less New Inventory	10,000	60,000
Gross Profit		$ 40,000
Selling Expenses	10,000	
Genl. and Adm. Expenses	10,000	20,000
Operating Profit		20,000
Interest and Discount		2,000
Net Profit		$ 18,000

Problems.

1. Divide 75 into two parts so that 4 times the lesser shall exceed the greater by 20.

2. Jones had 1/3 as much cash as Brown; afterwards Brown loaned $100 to Jones, who now had twice as much as Brown. How much had each at first?

3. Mary is 26 years of age, and Carroll is 36. How old was Mary when Carroll was twice her age?

4. A grocer sells a brand of coffee at 70 cents a pound, and another brand for 40 cents. How many pounds must he use of each to make a mixture of 50 pounds worth 49 cents per pound?

5. By the will of the deceased, the estate of $16,560, was to be divided among his wife, three sons, and three daughters. The wife was to receive four times as much as

each son, and each son three times as much as each daughter. How was the estate divided?

6. The sum of two consecutive numbers is 51. What are the numbers?

7. The sum of three consecutive odd numbers is 75. Find the numbers.

8. Three times a certain number added to six times the next consecutive number is equal to 42. What are the numbers?

9. When A was 18 years old, B was 12 years old. Five times A's present age equals six times B's present age. How old are A and B?

10. One number exceeds another by 3. If five times the smaller equals three times the larger, find the numbers.

11. Five times A's age two years ago equalled twice C's present age. C's present age is equal to five years more than twice A's present age. How old are A and C to-day?

12. One number is eight times as large as another number. If twice the sum of the two numbers is 72, find the numbers.

13. The sum of three consecutive numbers is 90. What are the numbers?

14. If John had two marbles fewer, he would have exactly twice as many as George has. Together they have 35 marbles. How many has each boy?

15. The unit's digit of a two-digit number is twice the tens digit. The sum of the digits is six. What is the number?

16. Twenty-six more than the product of the first two of four consecutive numbers is equal to the product of the third and fourth consecutive numbers. Find the numbers.

17. B's estate is equal to \$1000 less than twice A's estate. If \$3000 less than twice B's estate equals \$75,000, what are the estates of A and B worth?

18. C's estate is valued at \$1000 less than that of D, and E's estate is worth \$500 more than D's estate. Ten

times the value of C's estate equals five times the value of E's estate. Find the value of each of the estates.

19. A manufacturing concern wished to purchase a rectangular plot of ground upon which to build a plant. The plot of ground considered was twice as long as it was wide. If the width had been 25 feet less, it would have been three times as long as it was wide. If the selling price were \$45,000, what was the price per square foot?

20. The cash on hand of X concern was \$1000 more than their total expenses had been. Add the cash on hand and the amount of their accounts receivable, which equalled \$1000 more than twice their cash on hand; then deduct their expenses. If the remaining number is \$5000, what was the amount of their expenses, cash on hand, and accounts receivable at the end of the year?

Fractions.—In many accounting problems it is necessary to use simple fractions in algebraic form, such as "$\frac{1}{10}$ of x" or "$\frac{4}{5}$ times a," which when simplified are written $\frac{x}{10}$ or $\frac{4a}{5}$.

Example.—Divide 129 into two parts in such a manner that $\frac{2}{9}$ of one part shall equal $\frac{3}{8}$ of the other.

Solution.

Let $x =$ One part.

Then, $129 - x =$ The other part.

The problem states that: $\frac{2x}{9} = \frac{3}{8}(129 - x)$.

The common denominator of the fractions, as seen by inspection, is 72, and by the process of changing the fractions to a common denominator the result is: $\frac{16x}{72} = \frac{27}{72}(129 - x)$.

By multiplying each side of the equation by 72, to clear the equation of the fractions, the result is: $16x = 27(129 - x)$.

Eliminating the parentheses $16x = 3483 - 27x$.

Transposing, $43x = 3483$.

Dividing by 43, $x = 81$.

$129 - x = 48$.

Verification.

$\frac{2}{9}$ of $81 = 18$.
$\frac{3}{8}$ of $48 = 18$.

Example.—A superintendent of a certain plant is hired under a contract which provides that he is to receive ten per cent of the net profits of the business as a salary, after deducting his salary as an expense. The profits for the year were $13,200. Compute the salary.

Solution.

Let	s = the amount of salary.
The problem is stated:	$s = \frac{1}{10}(\$13200 - s)$.
Clearing of fractions by multiplication,	$10s = \$13200 - s$.
Transposing,	$11s = \$13200$.
Dividing by 11,	$s = \$1200$.

Verification.

Net profits	$13,200
Less salary	1,200
	$12,000
Dividing by 10 =	$ 1,200

Clearing of complex fractions.—To clear an equation of a complex fraction, when the denominator is a concrete number, it is usually better to multiply the opposite term of the equation by the denominator of the complex fraction than to use the complex fraction as a mixed number.

Example.

$$\frac{1 - \frac{1}{(1.06)^{10}}}{.06} = \$7.3600871.$$

Find the value of $(1.06)^{10}$ from the above equation.

Solution.

		Steps
	$\frac{1 - \frac{1}{(1.06)^{10}}}{.06} = \$7.3600871.$	(1)
Multiplying right-hand term by .06,	$1 - \frac{1}{(1.06)^{10}} = 7.3600871 \times .06.$	(2)
Clearing,	$1 - \frac{1}{(1.06)^{10}} = .4416052.$	(3)

Solution (Continued).

Transposing, $\frac{1}{(1.06)^{10}} = 1. - .4416052.$ (4)

Clearing, $\frac{1}{(1.06)^{10}} = .5583947.$ (5)

Dividing, $(1.06)^{10} = 1. \div .5583947.$ (6)

Clearing, $(1.06)^{10} = 1.790847.$ (7)

Problems.

1. What is the number, $\frac{1}{5}$ of which exceeds $\frac{1}{9}$ of it by 4?

2. Divide an estate of $18,000 so that the son's share will be $\frac{1}{2}$ that of the wife, and the daughter's share $\frac{3}{5}$ as much as that of the son.

3. Find the income of a manager who is to receive, in lieu of a salary, 12½% of the net profits of a concern, after deducting the salary. The profits before deducting the salary are $18,000.

4. Smith invests his money in three different classes of securities; in the first he invests $\frac{2}{5}$ of his money and it pays 6¼%; in the second he invests $\frac{2}{9}$ and it pays 5¼%; and in the third he invests the remainder, and it pays 3¾%. His total income is $2745. What is the value of his property?

5. At the end of the year, X concern found that twice their cash on hand equalled the amount of their accounts receivable which were six times the amount of their notes receivable. The total amount of their current assets was equal to fourteen times the amount of their notes receivable, or equal to $1000 less than four times their inventory. Find the value of each of their current assets.

Simultaneous equations with two or more unknowns.—When each of two equations contains two or more unknown quantities, and every equation may be satisfied by the same set of values for the unknown quantities, the equations are said to be *simultaneous*.

The value of the unknown quantities in two or more simultaneous equations sometimes may be found by combining them into a single equation, containing but one unknown quantity.

This combining may be done in several different ways, and is known as *elimination.*

The method of elimination by addition and subtraction is probably the most simple, and therefore will be used here.

It is often necessary to multiply one or sometimes both of the equations by such a number as will result in making one of the unknowns in each equation of equal absolute value.

Add or subtract the resulting equations, and the sum or remainder will result in an equation of one less unknown than the previous equations contained.

Example.—Solve for the values of a and b in the following simultaneous equations:

		Steps
	$10a - 6b = 38$	(1)
	$14a + 8b = 4$	(2)

Solution.

		Steps
Multiplying (1) by 7,	$70a - 42b = 266$	(3)
Multiplying (2) by 5,	$70a + 40b = 20$	(4)
Subtracting (4) from (3),	$-82b = 246$	(5)
Dividing by 82,	$-b = 3$	(6)
or	$b = -3$	(7)
Substituting the value of b in (1),	$10a + 18 = 38$	(8)
Transposing step (8),	$10a = 20$	(9)
Dividing by 10,	$a = 2$	(10)

Example.—A, as sales manager of a concern, is to receive a salary of 10% of the profits of a corporation, after deducting both the Federal income tax and the salary as expenses. The profits, before deduction of the tax and salary, were $22,000. Calculate the tax and salary.

Under the 1924 Income Tax Law a corporation is taxed at the rate of 12½% on net income, but is allowed an exemption of $2000 if the net income is less than $25,000.

Solution.

		Step
Let	$S =$ Salary.	(1)
And	$T =$ Tax.	(2)
	$S = 10\%\ (22{,}000 - T - S).$	(3)
Eliminating fractions,	$10S = 22{,}000 - T - S.$	(4)
Transposing,	$T + 11S = 22{,}000.$	(5)

Solution (Continued).

$$T = 12\tfrac{1}{2}\ (22{,}000 - 2000 - S). \quad (6)$$

Eliminating fraction, $8\,T = (22{,}000 - 2000 - S).$ (7)

Transposing, $8\,T + S = 22{,}000 - 2000.$ (8)

And, $8\,T + S = 20{,}000.$ (9)

Solving for S:

Multiplying (5) by 8, $8\,T + 88\,S = 176{,}000$ (10)

Subtracting (9), $8\,T + \quad S = \quad 20{,}000$ (11)

$87\,S = 156{,}000$ (12)

Dividing by 87, $S = \quad 1{,}793.10.$ (13)

Solving for T:

Multiplying (9) by 11, $88\,T + 11\,S = 220{,}000$ (14)

Subtracting (5), $T + 11\,S = \quad 22{,}000$ (15)

$87\,T \quad = 198{,}000$ (16)

Dividing by 87, $T \quad = \quad 2{,}275.86.$ (17)

Proof for Tax.

Net Profit	$22,000.00
Deducting Salary	1,793.10
Net Taxable Income	20,206.90
Less Specific Exemption	2,000.00
	18,206.90
Multiplying by 12½%	.12½
	$ 2,275.86

Proof for Salary.

Net Income		$22,000.00
Less Salary	$1,793.10	
" Tax	2,275.86	4,068.96
Subtracting		$17,931.04
Dividing by 10 for salary		$ 1,793.10

Example.—The following are the condensed balance sheets of three companies, who wish to know their true worth, as of December 31.

	Company A	*Company B*	*Company C*
Assets	$200,000	$500,000	$300,000
Investment in Company *B* (50%)	350,000		
Company *C* (20%)	250,000		
Investment in Company *C* (20%)		100,000	
Investment in Company *A* (10%)			100,000
	$800,000	$600,000	$400,000
Capital Stock	$500,000	$400,000	$300,000
Surplus	300,000	200,000	100,000
	$800,000	$600,000	$400,000

As each company owns stock in each of the others, there are three unknown quantities. The true value may be found as follows:

Solution.

SUMMARY OF OWNERSHIP

	Net Assets	*Company A*	*Company B*	*Company C*
Company *A* owns	$200,000		50%	20%
" *B* "	500,000			20%
" *C* "	300,000	10%		

Let A = Net worth of Company *A*.
" B = Net worth of Company *B*.
" C = Net worth of Company *C*.

Restating and solving:

				Steps
	$A - \frac{1}{2}B - \frac{1}{5}C =$		200,000	(1)
	$B - \frac{1}{5}C =$		500,000	(2)
	$-\frac{1}{10}A + C =$		300,000	(3)
Adding,	$\frac{9}{10}A + \frac{1}{2}B + \frac{3}{5}C =$		1,000,000	(4)
Multiplying (4) by 10,	$9A + 5B + 6C =$		10,000,000	(5)
" (2) by 5,	$+ 5B - C =$		2,500,000	(6)
Subtracting,	$9A + 7C =$		7,500,000	(7)
Multiply (3) by 7,	$-\frac{7}{10}A + 7C =$		2,100,000	(8)
Subtracting,	$9\frac{7}{10}A =$		5,400,000	(9)
Multiplying (9) by 10,	$97A =$		54,000,000	(10)
Dividing by 97,	$A =$		556,701.03.	(11)
Using (3) and substituting the value of $-\frac{1}{10}A$,	$-55{,}670.10 + C =$		300,000	(12)
Transposing,	$C =$		355,670.10.	(13)
Using (2) and substituting value of $-\frac{1}{5}C$,	$B - 71{,}134.02 =$		500,000	(14)
Transposing,	$B =$		571,134.02.	(15)

Verification.

	A	*B*	*C*	*Total*
Net Assets	$200,000.00	$500,000.00	$300,000.00	
A owns:				
½ of B.....	285,567.01			
⅕ of C.....	71,134.02			
Total	$556,701.03			$ 556,701.03
B owns:				
⅕ of C.....		71,134.02		
Total		$571,134.02		571,134.02
C owns:				
1/10 of A			55,670.10	
Total			$355,670.10	355,670.10
Total Value of Three Companies				$1,483,505.15

Problems.

1. Solve $15.917127 = \frac{1.79585635 - x}{.05}$.

2. Solve $6.632975 = \frac{4.265319 - x}{.04}$.

3. *A* and *B* are partners sharing profits in proportion to capital. Partnership profits were $1300. *A* owned ⅝ interest, plus $400, and his share of the profits was $600. What was the value of the business and the capital of each?

4. The American Manufacturing Company commenced business on January 1, 1918, with a paid-up cash capital equal to the sales for the year 1918. The net profits for the year 1918 were $26,100.

Of the total charges to manufacturing during the year, 40% was for materials, 30% for productive labor, and 30% for manufacturing expenses (including 5% depreciation on plant and machinery, amounting to $3,000).

The value of the materials used was 80% of the amount purchased, and 90% of the amount purchased was paid during the year.

The inventory value of finished goods on hand at December 31, 1918, was 10% of the cost of finished units delivered to the warehouse, and the work in process at that date was equal to 50% of the cost of units delivered to the warehouse.

The selling and administrative expenses were equal to 20% of the sales; also to 40% of the cost of goods sold. 90% of these expenses were paid during the year 1918. Plant and machinery purchased during the year were paid for in cash.

All labor and manufacturing expenses (exclusive of depreciation) were paid in full up to and including December 31, 1918.

Of the total sales for the year, 80% was collected and 1% charged off as worthless.

From the given data you are required to prepare a balance sheet and a profit and loss statement, showing cost of goods delivered to the warehouse, cost of goods sold, and net profits for the year.—*C. P. A.—Ohio.*

5. The Orinoco Coal Company was incorporated in 1904 with an authorized capital of $8,000.00 divided into eighty shares of the par value of $100 each, which were subscribed as follows:

Samuel Black	60 shares
William Green	10 shares
John White	10 shares

Samuel Black was elected president, George Brown vice-president and manager, and Charles Pinck secretary and treasurer. Neither Brown nor Pinck held any stock, but were to receive in addition to their salaries a percentage of the net profits, Brown 15 per cent, and Pinck 10 per cent.

No assessments whatever were made upon the stock, the credit to Capital Stock being offset by charge to Goodwill, and Franchises. The money necessary for the conduct of the business was borrowed on notes endorsed by the president. During the first two years the business was profitable; the borrowed capital was repaid and a surplus was accumulated. The third year the expense exceeded the profits and the company disposed of its stock of coal and furniture and fixtures, and discontinued business.

At a meeting of the stockholders the following balance sheet was presented:

Assets		*Liabilities*	
Cash	$ 31.12	Accounts Payable	$ 740.22
Accounts	817.32	Reserve for Bad Debts	376.05
Samuel Black	6,644.15	Capital Stock	8,000.00
Charles Pinck	2,264.14	Surplus	8,640.46
Goodwill and Franchises	8,000.00		
	$17,756.73		$17,756.73

It was announced that Treasurer Pinck, who was not financially responsible and was not bonded, had disappeared, and his account was uncollectible but would be reduced by crediting the account with his share of the net profits. President Black, whose account was also largely overdrawn, stated his willingness to undertake the work of liquidating the debts of the company, keeping the Reserve for Bad Debts as a protection against such of the accounts receivable as could not be collected, and offered to give his personal checks at once to William Green, John White, and George Brown for their

share of the surplus after charging to it the uncollectible portion of Pinck's account. The proposition was accepted.

Compute the bonuses to be credited to Pinck and Brown, and the remaining profits for the stockholders. Submit journal entries showing the disposition of the Surplus account of $8,640.46, and entries to record the personal payments made by Black. Show the balance sheet after this settlement.—*C. P. A.—Illinois.*

6. In closing the books of a Canning Company, you find the manager is to be paid upon a commission basis. His contract provides as follows:

> ". . . the party of the first part agrees to pay the party of the second part ten per cent of the net profits of each year as his salary. This commission to be computed on Net Profits after deducting bonus to Growers and Federal Income Tax. . . ."

You also find on the minutes of the directors the following statement:

> "If net profits, before bonus to Growers and Federal income tax is deducted, exceeds $20,000, then an amount shall be set aside out of profits for bonus to Growers equal to ten per cent of the net profits after deducting manager's salary, but before Federal income tax or bonus."

Your examination of the contract with Growers convinces you that the bonus is an allowable deduction for Federal tax purposes.

Net Profit, before deducting salary, bonus, or tax is $30,000. Compute necessary figures to properly close books. —*C. P. A.—Wisconsin.*

7. You have been engaged to audit the books and accounts of the *X* Corporation for the calendar year 1920. The factory and other accounting records are at *J;* the secretarial records are at *K.* The company's product is made to order only.

As a result of your audit at *J,* you have prepared the following:

CONDENSED PROFIT AND LOSS STATEMENT

FOR THE CALENDAR YEAR 1920

Gross sales (shipments)	$240,000.00
Less return sales and allowances	5,000.00
Net sales	235,000.00
Deduct:	
Cost of goods sold (shipped)	153,300.00
Gross profit on sales	81,700.00
Deduct:	
Selling and administrative expense	32,675.00
Net profit on operations	49,025.00
Add:	
Net financial income	975.00
Net profit for calendar year 1920	$ 50,000.00

Before the final preparation of your report, you go to *K* to examine the minutes of the board of directors. Upon your examination you find in the minutes, under date of March 12, 1920, the following:

> "Whereas, Mr. *A*, our factory superintendent, and Mr. *B*, our general sales manager, are devoting their entire time and attention to the affairs of the company and by their faithful devotion to its interests have increased the output of the company at the factory and the sales at *K*; and
>
> "Whereas, for this reason it is the sense of the directors that the said Mr. *A* and Mr. *B*, in addition to the amount of compensation regularly allowed them, should be further compensated;
>
> "Therefore be it resolved that fifty (50) per cent of the net operating profits of the *X* Company (after charging this additional compensation as an expense) arising from the net shipments over and above one hundred and seventy-five thousand dollars ($175,000) be divided equally between the said Mr. *A* and Mr. *B* as their additional compensation for the year."

Recast your profit-and-loss statement to give effect to the above additional information.

Show your procedure and give proof of your correctness. —*American Institute Examination.*

8. The following is a brief summary of the balance sheets of three companies at a given date:

	A	*B*	*C*
Total assets	$1,500,000	$2,000,000	$3,500,000
Total liabilities	$ 250,000	$ 750,000	$1,100,000
Capital stock (shares of $100 each)	1,000,000	750,000	2,000,000
Surplus	250,000	500,000	400,000
	$1,500,000	$2,000,000	$3,500,000

Each corporation owned part of the capital stock of the others, carried on their respective books as follows:

	Capital Stock In		
	A	*B*	*C*
A owned		15%	15%
Carried at		$100,000	$300,000
B owned	15%		10%
Carried at	$150,000		175,000
C owned	5%	5%	
Carried at	50,000	60,000	

The three companies agreed to consolidate and each to accept its pro rata share in the capital stock of a new corporation (*D*), having 1,000,000 shares of no par value.

What percentage of the shares of corporation *D* will the stockholders of *A*, *B* and *C* respectively receive and what will be the total equity or paid-in value of the consolidated capital? —*American Institute Examination.*

9. You are requested to find the value for consolidated balance sheet purposes of the following corporations. The separate balance sheets show the value of each corporation, as of December 31.

AMES COMPANY

Assets	$200,000	Liabilities	$100,000
Stock of Brown Co. (par)	30,000	Capital Stock	150,000
Stock of Colter Co. "	30,000	Surplus	10,000
	$260,000		$260,000

BROWN COMPANY

Assets	$100,000	Liabilities	$ 80,000
Stock of Colter Co. (par)	10,000	Capital Stock	50,000
Deficit	20,000		
	$130,000		$130,000

COLTER COMPANY

Assets	$100,000	Liabilities	$ 20,000
Stock of Brown Co. (par)	10,000	Capital Stock	50,000
		Surplus	40,000
	$110,000		$110,000

The holding company purchased the stock of the subsidiaries, paying book value therefor. Book values were par, for the subsidiaries had neither surplus nor deficit at the date of acquisition by the holding company.

10. Mr. F. B. Wilson is engaged by the Rochester Trust Association under contract, whereby he is to receive, in addition to a stipulated salary, 25% of the net profit over 10% on capital stock, after deducting income taxes and bonus. The amount of the capital stock is $100,000. Earnings for 1923, before considering tax and bonus, are $16,000. Compute tax and bonus.

11. The *A* Clinic has created an endowment fund to which it remits yearly 50% of the net profit, as "rent" which is to be considered an expense deduction. The profits for the year, before making the deduction for "rent," are $300,000. Compute the amount of remittance to be made to the fund.

12. Two concerns failed, owing each other money, the amounts of which were included in their respective Accounts Payable accounts.

A summary balance sheet of *X*, which is accepted as correct, is as follows:

BALANCE SHEET OF FIRM *X*

Due from *Y*	$ 10,000	Due to *Y*	$ 40,000
All other assets	180,000	All other liabilities	200,000
Deficit	50,000		
	$240,000		$240,000

A summary balance sheet of *Y*, which is accepted as correct, is as follows:

BALANCE SHEET OF FIRM *Y*

Due from *X*	$ 40,000	Due to *X*	$ 10,000
All other assets	160,000	All other liabilities	270,000
Deficit	80,000		
	$280,000		$280,000

The court holds that it is unfair to other creditors to allow such firms to strike a balance between their respective accounts, and then settle with outside creditors on the percentage basis. Accordingly, it is necessary to obtain the "ratio of insolvency," that is, the percentage (how many cents on the dollar) each concern will be able to pay on the basis of the respective balance sheets.

Determine this ratio of insolvency for each firm.—*C. P. A. New York.*

13. The General Machine Company and the Hamilton Machine Company are engaged in related lines of business. The former company has suffered a fire loss and its financial embarrassment has resulted in a receiver being appointed. The failure of this company has affected the financial position of the Hamilton Machine Company which is a creditor of the General Machine Company so it is declared bankrupt. The General Machine Company owes the Hamilton Machine Company $25,000.00 for capital stock issued to the General Machine Company. This stock was subsequently sold by the General Machine Company. The Hamilton Machine Company owes the General Machine Company $25,000.00 for Merchandise bought during September and October, 1924.

Accepting the following balance sheets as correct, calculate the percentage of claim each company can pay.

GENERAL MACHINE COMPANY

BALANCE SHEET

November 1, 1924

Assets		*Liabilities*	
Cash	$ 1,200.00	Notes Payable	$ 75,000.00
Accounts Receivable	55,000.00	Accounts Payable	256,200.00
Plant	250,000.00	Capital Stock	100,000.00
Deficit	125,000.00		
	$431,200.00		$431,200.00

HAMILTON MACHINE COMPANY

BALANCE SHEET

November 1, 1924

Assets		*Liabilities*	
Accounts Receivable	$ 75,000.00	Accounts Payable	$175,000.00
Buildings	20,000.00	Capital Stock	150,000.00
Machinery	55,000.00		
Deficit	175,000.00		
	$325,000.00		$325,000.00

The above statements show realizable value of all assets except the accounts between the two companies.—*C. P. A.—Wisconsin.*

Chapter 19

LOGARITHMS

Uses of logarithms.—It is not the purpose here to explain how logarithms are derived, but to make as clear as possible the simple use of logarithms by the accountant. The public accountant desires to know how to make a particular calculation with the greatest accuracy and in the least possible time. Many calculations can be made with comparative ease and a fair degree of accuracy by means of logarithms.

The use of logarithms greatly simplifies the *multiplying and dividing of numbers, the raising of numbers to powers, and the finding of roots of numbers.* Logarithms reduce multiplications and divisions to problems of addition and subtraction, the finding of the power of a number to a problem in multiplication, and the extraction of a root to a problem in division.

Exponents.—Mathematically speaking, "logarithms are exponents"; that is, the logarithm of a number is the exponent or power to which a base must be raised to produce a number.

To illustrate:

$$4^3 = 64.$$

or

$$4 \times 4 \times 4 = 64.$$

In the above,
4 = base.
3 = logarithm.
64 = number.

An exponent may be a fraction or a decimal, as:

$$10^{1.80618} = 64.$$

In this example,
10. = base.
1.80618 = logarithm.
64. = number.

Systems in common use.—Two systems of logarithms are in general use, the Common or Briggs' System, and the

Natural or the Naperian System. The base of the Common System is 10, and the base of the Natural System is an irrational number, $e = 2.71828$. In all the following work the Common System will be used.

In the writing of logarithms the base is seldom expressed, the logarithm only being written, as log 625 = 2.79588, or log 5 = 1.69897.

Parts of logarithms.—Every logarithm consists of two parts, the *characteristic* and the *mantissa*. The part to the left of the decimal point is the *characteristic,* and the part to the right of the decimal point is the *mantissa.*

The usual way of writing a logarithm of a number is similar to the following: Log 118. = 2.071882.

Sometimes a sign *"nl"* is used to separate a number from its logarithm and is read: 118. *nl* (the number whose logarithm is) 2.071882; or reversed it is read: 2.071882 *ln* (the logarithm of the number) 118.

Characteristic.—The *characteristic,* as has been stated before, is the number to the left of the decimal point of the logarithm. The characteristic of the logarithm of each number between 1 and 10 is 0.; between 10 and 100 is 1.; between 100 and 1000 is 2., and so forth. Also the characteristic of each number between .1 and 1 is — 1., and between .01 and .1 is — 2., and so forth.

Illustrations.

The logarithm of			1000.	has a characteristic of			3.		
"	"	"	100.	"	"	"	2.		
"	"	"	10.	"	"	"	1.		
"	"	"	1.	"	"	"	0.		
"	"	"	.1	"	"	"	— 1.	or 9. (mantissa)	— 10
"	"	"	.01	"	"	"	— 2.	or 8. "	— 10
"	"	"	.001	"	"	"	— 3.	or 7. "	— 10
"	"	"	.0001	"	"	"	— 4.	or 6. "	— 10
"	"	"	5.	"	"	"	0.		
"	"	"	25.	"	"	"	1.		
"	"	"	490.	"	"	"	2.		
"	"	"	370.	"	"	"	2.		
"	"	"	4.	"	"	"	0.		
"	"	"	2.7	"	"	"	0.		
"	"	"	.3	"	"	"	— 1.	or 9. (mantissa)	— 10
"	"	"	.49	"	"	"	— 1.	or 9. "	— 10

Positive characteristic.—It will be seen from the above list of numbers and of the characteristics of their logarithms,

that the characteristic is the same for the log of 490. as it is for the log of 370. Also it is the same for the log of 4. as it is for the log of 2.7. It is not the value of the digits in the number, but the number of places to the left of the decimal point in the number that gives the value to the positive characteristic. From the above the following rule may be inferred:

The characteristic of the logarithm of a number greater than one is one less than the number of places to the left of the decimal point.

Negative characteristic.—If a number is less than 1. its log has a *negative characteristic;* that is, the log of .3 or .49 has a negative characteristic of — 1., sometimes written "$\bar{1}$.", and the log of .01 or .0245 has a negative characteristic of — 2. If to any number is added a given quantity and from the sum is subtracted the same quantity, the value of the number has not been changed; that is, if 10 be added to 4, making 14, and from that sum 10 be subtracted, the value of the 4 is not changed, even though the expression may be written differently — as, 4. = 14. — 10. Therefore, instead of writing a logarithm with a negative characteristic as — 1.1959, it is found to be more convenient in some calculations to indicate the negative characteristic in the following manner, 9.1959 — 10. Or it may be more convenient in some cases to use a larger number; thus, — 1.1959 as 29.1959 — 30. By changing the form, the value has not been changed.

The characteristic of a number less than one is one more than the number of zeros between the decimal point and the first significant figure on the right.

Illustrations.

The log of .05 is — 2.6990
The log of .0005 is — 4.6990
The log of .000005 is — 6.6990

A characteristic may be either positive or negative, but a mantissa is *always* positive.

Mantissa.—The *mantissa* for the log of a number is a group of figures which stands for or represents the sequence of the digits of the number.

The mantissa for the log of 125. as shown by the table is .096910, which is also the mantissa of the logs of 12.5, 1.25, and 125000. *The mantissa depends on the sequence of the digits of a number,* and the *characteristic is used to show the correct placing of the decimal point.*

Illustration of Numbers and the Mantissas of Their Logarithms.

The	sequence	of	digits	in	125.	is	indicated	by	the	mantissa	.096910
"	"	"	"	"	1250.	"	"	"	"	"	.096910
"	"	"	"	"	125000.	"	"	"	"	"	.096910
"	"	"	"	"	12.5	"	"	"	"	"	.096910
"	"	"	"	"	1.25	"	"	"	"	"	.096910
"	"	"	"	"	.125	"	"	"	"	"	.096910
"	"	"	"	"	475.	"	"	"	"	"	.676694
"	"	"	"	"	625.	"	"	"	"	"	.795880
"	"	"	"	"	920.	"	"	"	"	"	.963788
"	"	"	"	"	1.	"	"	"	"	"	.000000

Illustration of Numbers Corresponding to the Complete Logarithm (Characteristic Plus Mantissa).

Log	1235.	= 3.09167
"	123.5	= 2.09167
"	12.35	= 1.09167
"	1.235	= 0.09167
"	.1235	= — 1.09167
"	.01235	= — 2.09167
"	.001235	= — 3.09167

The table given in the appendix of this book is called a six-place table. This means that the mantissas as given are accurate to the sixth place. It does not, necessarily, mean that by using a six-place table one can perform calculations accurately to the sixth place. There are tables of logarithms accurate to six, eight, or even ten places, but it is thought that the table as given is sufficient for illustration. If the accountant has many computations involving large numbers, it is advisable to procure a more extended table.

How to use a table of logarithms.—To find the logarithm for numbers of one significant figure.—The table shows only the mantissa of the logarithm. If it is desired to find the mantissa of a number such as 2, 20, 200, 2000, or any number whose only significant figure is 2, it is necessary to turn to the table and in the column at the left, headed

"n", run down the line until the number 200 is found. To the right of this number, in the column headed "0", the mantissa ".301030" is found. This is the mantissa for the logs of 2, 20, 200, or .2, .02, .002, and so forth. The mantissa for any other number of one significant figure may be found in a similar manner. In order to get the complete logarithm of a number it is necessary to supply the characteristic. The characteristic of the log of 2 is "0". Hence, the complete logarithm of 2 is "0.301030".

For numbers of two significant figures.—If it is desired to find the logarithm of a number containing two significant figures, as 17, 170, or 1.7, it is necessary to look in the column at the left of the table headed "n", and run down the column until 170 is found. In the column to the right of 170 headed "0" the mantissa .230449 is found. Keep in mind that this is a mantissa for the logs of 17, 170, 1700, and so forth.

For numbers of three significant figures.—Assume that the logarithm of 118 is desired. In the left-hand column of the table headed "n" find the number 118. To the right of this number the mantissa .071882 is given. This, of course, is the mantissa for the logs of 118, 1.18, 11800, and so forth. In the foregoing illustrations, the mantissa only was found. By the rules previously given in regard to the characteristic, it is found that the characteristic of the log of 118 is 2. Therefore, the logarithm of the number 118 is 2.071882.

For numbers of four significant figures.—To illustrate: Find the logarithm of 1648. As previously stated, find the number 164 at the left of the table. On the line horizontally to the right of 164 in the column headed "8", the mantissa found is .216957. The characteristic, as previously explained, for the log of 1648 is 3. The complete logarithm of 1648 is 3.216957.

Interpolation for numbers of five or more significant figures.—The logarithm of a number of five places may be found by the process of interpolation. This is a computation on the assumption that the differences of the mantissas are proportional to the differences in the numbers given. This proportion is not strictly accurate, but is sufficiently so for most purposes.

To illustrate.—Find the logarithm of 131525. The *difference* between the mantissas of the log of 1315 and the log of the next higher number, 1316, as shown by the table in the column headed "d" on the right-hand side opposite 131, is 330; this should be more correctly stated as .000330. The excess of the 131525 over 131500 is $^{25}/_{100}$ of the difference between 131500 and 131600. Therefore, multiply .000330 by $^{25}/_{100}$ to obtain the fractional part of the difference in the mantissa, which is .0000825. The mantissa for 131500 is .118926. To this add the .0000825, which will give the mantissa for 131525. or .1190085. Stated again,

The log of 131600. has a mantissa of .119256
" " " 131500. " " " " .118926
The difference of 100 = the difference of .000330
131525 — 131500 = 25.

The difference of 25 in the numbers should have a proportional part of the difference in the mantissas, .000330, added to the mantissa of the log of the smaller number.

$^{25}/_{100}$ of .000330 = .0000825.
.118926 + .0000825 = .1190085 or approximately the correct mantissa.

The characteristic for the log of 131525 is 5.
Therefore the complete log of 131525. is 5.1190085.

Problems.

Find the logarithms of the following:

1.	6.	5.	24.83	9.	34,976.5	13.	102.4
2.	95.	6.	475.6	10.	125.	14.	6.462
3.	.085682	7.	.0998	11.	.89	15.	.037
4.	4.32	8.	1275.	12.	375.5	16.	1.03

To find a number when the logarithm is given.—If the above process of finding the logarithm from a number is reversed, the number can easily be found from the logarithm. This process is called finding the *antilogarithm*. It is necessary first to find the digits of the number, and this must be done from the mantissa. If the mantissa can be found in the table, take the digits corresponding to the mantissa, and point off these digits decimally as indicated by the characteristic.

To illustrate.—The logarithm is 0.281033. By referring to the table, the mantissa, .281033, is found in the vertical column headed "0", opposite the number 191. This indicates that the sequence of digits, ignoring possible initial or final zeros, is 191. Using the characteristic for the correct placing of the decimal, the number is found to be 1.91.

To find a number whose mantissa is not in the table.—If the mantissa is not given in the table, it is necessary to reverse the process of interpolation.

To illustrate.—5.1190085 is the logarithm of what number?

By referring to the table, it is found that the mantissa for the number nearest the one given and less than it, is .118926. By subtracting this from the mantissa given in the problem, the difference is found to be .0000825. Use this as the numerator, and the difference, .000330, as indicated by the table in the column at the right, as the denominator. By reducing this fraction $\frac{.0000825}{.0003300}$, the result is found to be .25, or $^{25}/_{100}$ of the difference between 1315 and 1316. By adding this amount to 1315, the sequence of digits is found to be 131525, which when pointed off, as indicated by the characteristic, gives the result, 131525.

Problems.

What are the numbers represented by the following logarithms?

1.	4.575225	4.	— 1.391468	7.	2.942292	10.	3.105542
2.	0.778262	5.	— 2.096926	8.	— 2.574665	11.	0.635553
3.	3.632587	6.	1.844334	9.	— 3.882872	12.	— 2.932525

Rules for computation by logarithms.—

Rule 1.—To multiply numbers, find the logarithm of each number, add the two logarithms, and the sum will be the logarithm of the product.

Rule 2.—To divide numbers, find the logarithm of the divisor and dividend, subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient.

Rule 3.—To obtain a power of a number, find the logarithm of the number and multiply this logarithm by the exponent of the power sought. The product will be the logarithm of the power of the number.

Rule 4.—To obtain a root of a number, find the logarithm of the number, divide the logarithm by the index of the root sought, and the quotient will be the logarithm of the root of the number.

Multiplication by logarithms.—Use Rule 1 to solve the following problem: Multiply 635 by 22.

Solution.

Log 635 = 2.802774
Log 22 = 1.342423
Log of product = 4.145197

In the table, the mantissa .145196 + is found to correspond to the digits 1397. The characteristic 4 indicates that the product has five digits to the left of the decimal point, or that the product is 13970. (Correct to five places only.)

Problems.

Multiply:

1. 875. × 125.
2. 34.7 × 4.25
3. 927. × 83.47
4. 2.7 × 6.2
5. 42. × .09
6. 6.24 × .0037
7. .649 × 84.5
8. 896. × 62.
9. 3.75 × 9.75
10. 89.6 × .375
11. 3.4 × .46
12. .725 × 1.35
13. 2.96 × .826
14. .76 × 28.
15. 4.9 × .92
16. 91. × .7854

Division by logarithms.—Rule 2 gives the procedure for division by logarithms.

Division of a greater number by a lesser number.—Divide 875 by 37.

Solution.

Log 875 = 2.942008
Log 37 = 1.568202
Log quotient = 1.373806

By reference to the table, it is found that the mantissa, .373806, corresponds to the digits, 23648. The characteristic indicates that the quotient has two digits to the left of the decimal place. Pointed off, the answer is 23.648.

Division of a lesser number by a greater number.—At times it is necessary to divide a smaller number by a larger one, which, of course, gives a decimal number for the quotient. This requires the operation of *subtraction* of a greater logarithm from a lesser one. To do this, change the form of the logarithm of the minuend; that is, add 10 (or a multiple of 10) to the characteristic of the minuend, and write a — 10 (or a multiple of — 10, the same multiple to be used in each case) after the mantissa.

To illustrate.—Divide 269 by 239,000.

Solution.

Log 269 = 2.429752, or changed is	12.429752 —10
Deduct log 239,000 or	5.378398
Log quotient	= 7.051354 —10

By changing the form of 7.051354 — 10 to its simple form, it becomes — 3.051354. By reference to the logarithm table of mantissas, and by applying the rules for pointing off numbers by the characteristic, the quotient is found to be .0011255. (Correct to five significant figures.)

The above applies in the subtraction of a negative logarithm from a positive one.

To illustrate.—Divide 14,200 by .000191.

Solution.

Log 14,200. = 4.152288, changed	= 14.152288 —10
Log .000191 = —4.281033, "	= 6.281033 —10
Log quotient	= 7.871255 — 0

By finding the antilog of 7.871255, the resulting quotient is ascertained to be 74,345,500. (Correct to five significant places.)

Problems.

Divide:

1.	879.	by	2.35	11.	376.	by	.125
2.	8.56	"	.92	12.	64.	"	800.
3.	9.27	"	7.5	13.	34.7	"	24.7
4.	.327	"	9.2	14.	.296	"	269.
5.	.27	"	.023	15.	87.2	"	936.
6.	97600.	"	.475	16.	2.	"	402.5
7.	.3475	"	.49	17.	.01	"	346.27
8.	24.	"	.0004	18.	1.	"	1.0606
9.	2.9	"	.4280	19.	1.	"	.9764
10.	42.	"	.42	20.	10.	"	3.1416

Powers of numbers.—Rule 3 gives the procedure for finding the powers of numbers.

Example.—Find the fourth power of 26.

Solution.

Log 26	= 1.414973
Multiply by the exponent of the power,	4
Log power	= 5.659892

By referring to the table, it is found that the mantissa .659892 represents the sequence of digits, 456975. The characteristic 5 indicates that the product has six digits to the left of the decimal point. Hence, the fourth power of 26 is 456,975. (Correct to five significant places.)

Process with a negative characteristic.—

Example.—Find the fifth power of .025.

Solution.

Log .025 = — 2.397940, or changed	= 8.397940 —10
Multiply by the exponent of the power	5
	41.989700 —50
41.9897 —50, changed	= — 9.9897
The mantissa .9897	= 976562
Pointed off by the characteristic	= .00000000976562

Problems.

1.	Find the 9th power of 65.	5.	Find the 6th power of 9200.
2.	" " 5th " " 23.	6.	" " cube of 12.7
3.	" " 8th " " 96.	7.	" " cube of 18.
4.	" " 4th " " 15.	8.	" " square of 37.82

Compute the following:

9.	6^3	12.	1.035^{24}	15.	1.06^{28}	18.	2.12^{13}
10.	150^6	13.	125^5	16.	$.97^{36}$	19.	$.732^{25}$
11.	1.04^{40}	14.	$.57^3$	17.	$.07^{25}$	20.	1.02^{45}

Roots of numbers.—Rule 4 gives the procedure for finding roots of numbers.

Example.—Find the cube root of 875.

Solution.

Log 875	= 2.942008
Divide by index of the root,	3) 2.942008
Log quotient	= 0.980669 +

By referring to the table, it is found that the mantissa, .980669, corresponds to the sequence of digits, .95646, and the characteristic indicates that the number is 9.5646.

Process with negative characteristics.—In stating the equivalent of the negative characteristic of a logarithm, care must be taken to see that the right-hand number or minus quantity is exactly divisible by the index of the root.

Example.—Find the 4th root of .125.

The negative characteristic of the complete logarithm of .125 may be stated in different ways, as, Log .125 = — 1.096910, 3.096910 — 4, 7.096910 — 8, 19.096910 — 20, or 39.096910 — 40.

In this problem, in order to avoid any complication, it is best to have the right-hand number of the characteristic exactly divisible by 4 (the index of the required root).

Solution.

Log .125	=	— 1.096910 or,
Log .125	=	39.096910 — 40
Divide by the index of root,		4) 39.096910 — 40
Log quotient	=	9.774228 — 10
Or changed	=	— 1.774228

The mantissa .774228 corresponds to the succession of digits 594604. (Accurate to five places.) Pointed off by the characteristic, — 1. the 4th root of .125 is .594604.

Problems.

1.	Find the square root of	81.	6.	Find the 4th root of	.347
2.	" " " " "	125.	7.	" " 9th " "	1.25
3.	" " cube " "	900.	8.	" " 6th " "	.9873
4.	" " 6th " "	750.	9.	" " 5th " "	5.878
5.	" " 10th " "	10.	10.	" " 63rd " "	.875

Extract roots as indicated:

11. $\sqrt[3]{1728.}$
12. $\sqrt[3]{144.}$
13. $\sqrt[3]{81.}$
14. $\sqrt[4]{367.}$
15. $\sqrt[3]{68.}$
16. $\sqrt[7]{3.65}$
17. $\sqrt[10]{2.48}$
18. $\sqrt[3]{.846}$
19. $506 \times \sqrt[4]{40}$
20. $\sqrt[5]{(381)^3}$
21. $(126)^4 \times \sqrt[4]{\frac{81}{187}}$
22. $(9.78)^5 \times \sqrt[3]{\frac{99}{108}}$

Chapter 20

LOGARITHMIC CHARTS

BY EARL G. LEE

Charting.—A textbook on business mathematics would not be complete without some reference to charts, which are becoming increasingly popular as a means of presenting the results of accounting and mathematical computations. To attempt, however, even an outline of all the various forms of graphic presentation which have been adapted to commercial purposes alone, to say nothing of their use in engineering and scientific work, would require a disproportionate amount of space, and the result would prove inadequate.

The student is therefore referred to standard textbooks on the subject, and attention here is directed merely to a single type of chart. This type is selected because it is the development of a mathematical principle.

The common logarithm of the Briggs' table, being the expression of any number in terms of the power of ten, is particularly adapted to the presentation of percentage relationships. This fact has led to the construction of a chart design in which percentage relationships are revealed by a comparison of different sets of data plotted in terms of numerical magnitude.

The logarithmic chart.—The logarithmic chart is a variation of the rectilinear type. The ruling differs from that of the customary co-ordinate chart in that the data lines representing the scale are not evenly spaced, but conform to certain proportions expressed by the first ten numbers in a table of common logarithms.

Figure *A* illustrates the method of laying out such a scale. The first column of figures is purely for draughting purposes, and consists of the first ten figures of a logarithmic table, only the first two decimal digits being used in each case.

In laying out the vertical scale, each horizontal line is spaced a distance from the base of the scale represented by the proportion which its scale number bears to 100; that is, the first line above the base line is thirty one-hundredths of the total height of the scale, the next line is forty-seven one-hundredths, and so on, the top line representing one hundred. A ruler having a 100-millimeter scale may be conveniently used for the purpose.

The second column represents the numerical magnitude scale, and is numbered from one to ten, or some multiple thereof, the ruled lines being spaced according to the logarithms of these numbers. The result is that when data are plotted on such a chart in terms of numerical magnitude, the relationships between the various groups of data plotted will be correct percentage relationships. This does not hold true where data are plotted numerically on an ordinary co-ordinate chart; and where the curves represent widely differing magnitudes, an attempted comparison of the fluctuations in the data will be misleading. In order that data plotted on a co-ordinate chart may present a correct percentage relationship, the lines must represent a percentage, not a numerical scale.

In comparing data of widely differing magnitudes on a logarithmic scale, more than one grouping of logarithmic rulings must be used. Each such grouping is called a cycle, because it represents ten, or some multiple thereof. For instance, in comparing data represented by figures in the hundreds group with data running into the thousands, two cycles would have to be used, and if into the ten-thousands, three cycles.

Ratio charts.—Figure *B* illustrates a two-cycle chart. Only the horizontal lines are ruled in accordance with the logarithmic scale. The vertical lines are evenly spaced. This is called a semi-logarithmic or ratio-ruled chart, and is the kind most commonly used.

The use of the full logarithmic chart, with the ratio ruling in both directions, is rare, and is confined chiefly to mathematical demonstrations.

The ratio chart has its limitations. While it exhibits correct percentage relationships, it does not record correct

numerical magnitudes, and it should be used only where percentage relationships are desired. But because relationships between fluctuating data are more easily grasped in terms of percentages than in purely numerical terms, the semi-logarithmic chart has a wide field of usefulness.

It will be noticed that the base line of logarithmic charts is numbered 1, instead of 0, because in the tables the log of 1 is .0. Therefore, in a logarithmic chart there is no zero line.

All rectilinear charts have two axes, the horizontal, called the *x*-axis, and the vertical, the *y*-axis.

To illustrate the use of a full logarithmic chart, a simple example in multiplication may be cited. Applying the principle that in multiplication the logarithm of the product of two numbers is the sum of the logarithms of the numbers, the addition may be performed graphically on a logarithmic chart. By doubling the distance of any number represented by a digit, the square of the digit is obtained. The distance from 1 to 9 is twice the distance from 1 to 3, because 9 is the square of 3.

As the scales of both axes are the same, lines projected at right angles from identically numbered points on both axes will complete a square, the diagonal of which is 45 degrees. Therefore, if a line is projected upward at an angle of 45 degrees, to the right from any point on the base of the *x*-axis, for instance, any horizontal line which intersects this diagonal will, if projected downward from the intersecting point to the base line of the *x*-axis, and at right angles thereto, record on the *x*-axis digit scale the product of the two numbers representing the starting points of the diagonal and the horizontal lines.

Figure *A* illustrates a computation of this kind, and affords a mechanical demonstration of the logarithmic principle referred to. A diagonal line is projected upward at an angle of 45 degrees from Figure 2 on the *x*-axis. The point at which it intersects the horizontal line numbered 4 on the *y*-scale is directly above Figure 8 on the *x*-scale, and two times four is eight. Likewise the sum of the distances, 1 to 2 on the *x*-axis, and 1 to 4 on the *y*-axis, will, if laid off by a pair of dividers, arrive at point 8 on either scale.

Example.

CHARTED IN FIGURE *B*

	Sales	*Cost*	*Gross Profit 40%*	*Expense*	*Net Profit*
January	$ 5,400	$ 3,240	$ 2,160	$ 2,000	$ 160
February	4,500	2,700	1,800	1,500	300
March	8,000	4,800	3,200	2,200	1,000
April	10,000	6,000	4,000	2,500	1,500
May	7,000	4,200	2,800	2,100	700
June	6,000	3,600	2,400	1,500	900
	$40,900	$24,540	$16,360	$11,800	$4,560

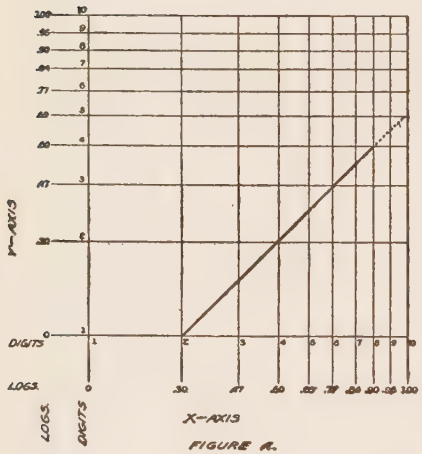


FIGURE A.

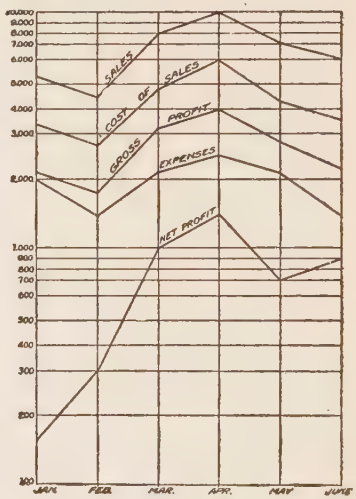


FIGURE B.

Problems.

1. Chart the following in such a manner as to show the results to advantage:

Month	*Pairs of Shoes Manufactured*	*Cost of Sales*	*Sales*	*Gross Profit*
January	3,200	$ 4,250	$ 5,400	$1,150
February	2,500	4,050	4,500	450
March	3,600	4,700	4,800	100
April	3,700	4,850	5,200	350
May	3,100	4,800	6,000	1,200
June	3,000	4,900	5,400	500
	19,100	$27,550	$31,300	$3,750

2. Chart the following condensed operating statements:

	1917	*1918*	*1919*	*1920*	*1921*
Sales	$175,000	$180,000	$200,000	$210,000	$250,000
Cost of Sales	132,000	130,000	150,000	180,000	190,000
Selling Expense	20,000	12,000	14,000	10,000	15,000
Administration Exp.	15,000	15,500	16,000	16,000	18,000
Net Profit	xxx	xxx	xxx	xxx	xxx

3. Chart in Logarithmic form:

Date	*Labor Hours*	*Amount*	*Rate per 100 Hours*
1923			
Jul. 7	4275	737.44	17.25
" 14	4625	809.38	17.50
" 21	4575	790.56	17.28
" 28	4852	846.67	17.45
Aug. 4	4965	874.83	17.62
" 11	4872	851.14	17.47
" 18	4775	853.29	17.87
" 25	4825	861.26	17.85
Sep. 1	4650	832.35	17.90
" 8	4652	860.62	18.50
" 15	4595	852.37	18.55
" 22	4500	837.00	18.60
" 29	4810	897.07	18.65
Oct. 6	4825	907.10	18.80
" 13	4950	950.40	19.20
" 20	4685	913.58	19.50
" 27	4850	950.60	19.60

4. Construct a chart based on the data given below:

The earnings of the Oregon-Washington Railroad and Navigation Company are included in those of the Union Pacific Railroad Company which, for the fifteen years ending December 31, 1920, are as follows:

	Gross Earnings from Operation	*Income Available for Bond Interest*	*Bond Interest*
1920	$209,049,510	$48,261,218	$15,419,279
1919	177,447,698	49,575,786	15,046,284
1918	158,845,176	47,104,102	14,396,337
1917	130,101,864	55,379,230	13,842,730
1916	114,412,607	57,009,606	13,826,197
1915	91,952,245	45,390,006	13,803,767
1914	91,845,305	47,168,025	14,062,108
1913	91,810,307	50,990,743	14,201,658
1912	84,180,097	48,120,710	14,068,704
1911	87,201,971	52,611,056	12,623,282
1910	88,506,465	57,976,648	12,455,577
1909	77,360,429	54,941,782	13,331,368
1908	74,422,777	46,977,313	11,245,899
1907	74,656,152	33,347,398	8,640,608
1906	67,281,543	30,305,756	8,870,898

Chapter 21

PROGRESSION

Definition.—A progression is a series of numbers, each term of which is obtained from the preceding or following term by a fixed law; as 2, 4, 6, 8, and so forth, or 2, 4, 8, 16, and so forth.

Increasing series.—A progression, each term of which is greater than the preceding term, is known as an increasing or ascending series.

Decreasing series.—A progression, each term of which is less than the preceding term, is known as a decreasing or descending series.

Arithmetical progression.—When each term of a progression differs from the preceding or following term by a common difference, it is said to be an arithmetical progression.

Symbols.—The five elements of an arithmetical progression are represented by certain well-established symbols, which are:

a = First term.
l = Last term.
n = Number of terms.
d = Common difference.
s = Sum of the terms.

Relation of elements.—The five elements whose symbols have just been given are so related, that if any three of them are given, the remaining two may be found.

Increasing series.—The illustrations given below are for the purpose of showing the formulas and solutions of problems in connection with increasing series using the term values as follows:

Last term	= 35.
Sum	= 222.
First term	= 2.
Common difference	= 3.
Number of terms	= 12.

To find the last term.—

Algebraic Formula 1. $a + (n-1)\,d = l.$

Arithmetical Substitution. $2 + (12-1)\,3 =$ Last term.

Solution.

Removing parentheses, $12 - 1 = 11.$
Multiplying by 3, $11 \times 3 = 33.$
Adding, $2 + 33 = 35$ or last term.

To find the sum.—

Algebraic Formula 2. $\frac{n}{2}(a + l) = s.$

Arithmetical Substitution. $\frac{12}{2}(2 + 35) =$ Sum.

Solution.

Adding the terms in parentheses, $2 + 35 = 37.$
Multiplying by the fraction, $37 \times \frac{12}{2} = 222,$ or Sum.

To find the first term.—

Algebraic Formula 3. $l - (n-1)\,d = a.$

Arithmetical Substitution. $35 - (12-1)\,3 =$ First term.

Solution.

Removing parentheses, $12 - 1 = 11.$
Multiplying by 3, $11 \times 3 = 33.$
Subtracting, $35 - 33 = 2,$ or first term.

To find the common difference.—

Algebraic Formula 4. $\frac{l-a}{n-1} = d.$

Arithmetical Substitution. $\frac{35-2}{12-1} =$ Common difference.

Solution.

Simplifying the numerator, $35 - 2 = 33.$
Simplifying the denominator, $12 - 1 = 11.$
Dividing, $33 \div 11 = 3,$ or Common difference.

To find the number of terms.—

Algebraic Formula 5. $\frac{l-a}{d} + 1 = n.$

Arithmetical Substitution. $\frac{35-2}{3} + 1 =$ Number of terms.

Solution.

Simplifying the numerator,	$35 - 2 = 33.$
Dividing by denominator,	$33 \div 3 = 11.$
Adding,	$11 + 1 = 12,$ or Number of terms.

Decreasing series.—The decreasing series will be illustrated in the same manner as the increasing series, with term values as follows:

Last term	$= -21.$
Sum	$= -56.$
First term	$= 7.$
Difference	$= -4.$
Number of terms	$= 8.$

To find the last term.—

Algebraic Formula 1.	*Arithmetical Substitution.*
$a + (n-1)\,d = l.$	$7 + [(8-1)\,(-4)] =$ Last term.

Solution.

Removing parentheses,	$8 - 1 = 7.$
Multiplying by -4,	$[7\,(-4)] = -28.$
Subtracting,	$7 - 28 = -21.$

To find the sum.—

Algebraic Formula 2.	*Arithmetical Substitution.*
$\frac{n}{2}\,(a + l) = s.$	$\frac{8}{2}\,(7 + -21) =$ Sum.

Solution.

Simplifying the fraction,	$\frac{8}{2} = 4.$
Adding,	$(7 + -21) = -14.$
Multiplying,	$4 \times -14 = -56.$

To find the first term.—

Algebraic Formula 3.	*Arithmetical Substitution.*
$l - (n-1)\,d = a.$	$-21 - [(8-1)\,(-4)] =$ First term

Solution.

Removing the parentheses,	$(8-1) = 7.$
Multiplying,	$[7\,(-4)] = -28.$
Simplifying,	$(-21) - (-28) = (-21) + 28.$
	$(-21) + 28 = 7.$

To find the common difference.—

Algebraic Formula 4. $\dfrac{l-a}{n-1}=d.$

Arithmetical Substitution. $\dfrac{-21-7}{8-1}=$ Common difference.

Solution.

Simplifying the numerator, $-21-7=-28.$
Simplifying the denominator, $8-1=7.$
Dividing, $-28\div+7=-4.$

To find the number of terms.—

Algebraic Formula 5. $\dfrac{l-a}{d}+1=n.$

Arithmetical Substitution. $\dfrac{(-21)-7}{-4}+1=$ Number of terms.

Solution.

Simplifying the numerator, $(-21)-7=-28.$
Dividing by the denominator, $(-28)\div(-4)=7.$
Adding, $7+1=8,$ number of terms.

Problems.

1. Given $a=1$, $n=8$, $l=15$; find d and s.

2. Given $n=12$, $l=-17$, $s=-72$; find d and a.

3. Given $l=57$, $n=23$, $a=-9$; find d and s.

4. Find the sum of all the even numbers from 2 to 100, inclusive.

5. Find the sum of the numbers on the face of a clock.

6. The first term of a progression is 6, the number of terms is 25, the common difference is 7; find the last term and the sum.

7. The first term is 6, the last term is 195, and the common difference is 7; find the number of terms.

8. A man invests his savings in the shares of a building and loan association, depositing $500.00 the first year. At the beginning of the second year he is credited with $30 interest and deposits $470. At the beginning of the third

year he is credited with $60 and deposits $440. What is his credit at the end of 10 years, and how much cash has he paid in?

9. A city floats a bond issue of $50,000, bearing interest at 5%, and the principal is to be repaid in 20 equal annual installments. Compute the amount of interest that the city will pay during the life of the bonds.

10. If you should deposit $1 the first week of January, $2 the second week, $3 the third week, and so on for the 52 weeks in the year, how many dollars would you have to your credit, excluding interest accumulations?

11. An employee started work for a company at $960 for the first year with a regular yearly increase of $120. What was his salary 12 years later? How much did the company pay him in the course of the 12 years?

Geometrical progression.—A geometrical progression is one in which any term after the first is obtained by multiplying the preceding term by a fixed number known as the ratio.

Elements.—In geometrical progression, there are five elements so related that any three being given, the others may be found. These five elements, together with their symbols, are:

First term	$= a.$
Last term	$= l.$
Ratio	$= r.$
Number of terms	$= n.$
Sum of the series	$= s.$

Increasing series.—As in arithmetical progression, the formulas and solutions will be based on one set of terms, and the solutions will be given as though the required term in the example in question were lacking. In some cases, two different formulas may be used.

First term	$= 3$
Ratio of increase	$= 2$
Number of terms	$= 6$
Last term	$= 96$
Sum of terms	$= 189$

To find the last term.—

Algebraic Formula 1. $a\,(r)^{n-1} = l.$

Arithmetical Substitution. $3\,(2)^{6-1} =$ Last term.

Solution.

Simplifying the exponent, $(2)^{6-1} = (2)^5.$
Finding the power, $(2)^5 = 32.$
Simplifying, $3 \times 32 = 96,$ or Last term.

To find the sum.—

Algebraic Formula 2. $\dfrac{lr - a}{r - 1} = s.$

Arithmetical Substitution. $\dfrac{(96 \times 2) - 3}{2 - 1} =$ Sum.

Solution.

Simplifying, $(96 \times 2) = 192.$
Subtracting, $192 - 3 = 189.$
Simplifying the denominator, $2 - 1 = 1.$
Dividing, $189 \div 1 = 189,$ or Sum.

To find the first term.—

Algebraic Formula 3. $\dfrac{l}{(r)^{n-1}} = a.$

Arithmetical Substitution. $\dfrac{96}{(2)^{6-1}} =$ First term.

Solution.

Simplifying the exponent, $(2)^{6-1} = (2)^5.$
Raising to power, $(2)^5 = 32.$
Dividing, $96 \div 32 = 3,$ or First term.

To find the ratio.—

Algebraic Formula 4. $\sqrt[n-1]{\dfrac{l}{a}} = r.$

Arithmetical Substitution. $\sqrt[6-1]{\dfrac{96}{3}} =$ Ratio.

Solution.

Simplifying the exponent, $6 - 1 = 5.$
Simplifying the fraction, $96 \div 3 = 32.$
Extracting the 5th root, $\sqrt[5]{32} = 2,$ or Ratio.

Decreasing series.—

$a = 96.$
$n = 6.$
$r = \frac{1}{2}.$
$l = 3.$
$s = 189.$

To find the last term.—

Algebraic Formula 1. $a\,(r)^{n-1} = l.$

Arithmetical Substitution. $96\,(\frac{1}{2})^{6-1} =$ Last term.

Solution.

Simplifying the exponents, $6 - 1 = 5.$
The value of: $(\frac{1}{2})^5 = \frac{1}{32}.$
Multiplying, $96 \times \frac{1}{32} = 3,$ Last term.

To find the sum.—

Algebraic Formula 2. $\dfrac{lr - a}{r - 1} = s.$

Arithmetical Substitution. $\dfrac{3 \times \frac{1}{2} - 96}{\frac{1}{2} - 1} =$ Sum.

Solution.

Multiplying, $3 \times \frac{1}{2} = \frac{3}{2}.$
Subtracting, $\frac{3}{2} - 96 = -\frac{189}{2}.$
Simplifying the denominator, $\frac{1}{2} - 1 = -\frac{1}{2}.$
Dividing, $(-\frac{189}{2}) \div (-\frac{1}{2}) = 189,$ or Sum.

To find the first term.—

Algebraic Formula 3. $\dfrac{l}{r^{n-1}} = a.$

Arithmetical Substitution. $\dfrac{3}{(\frac{1}{2})^{6-1}} =$ First term.

Solution.

Simplifying the denominator, $(\frac{1}{2})^{6-1} = \frac{1}{32}.$
Dividing, $3 \div \frac{1}{32} = 96,$ or First term.

To find the ratio.—

Algebraic Formula 4. $\sqrt[n-1]{\dfrac{l}{a}} = r.$

Arithmetical Substitution. $\sqrt[6-1]{\dfrac{3}{96}} =$ Ratio.

Solution.

Simplifying the fraction, $\frac{3}{96} = \frac{1}{32}.$
Simplifying the radical, $\sqrt[5]{\dfrac{1}{32}} = \frac{1}{2},$ or Ratio.

Progression problems solved by the use of logarithms.—Progressions deal with increases and decreases in proportional ratios.

Example.—What is the average yearly rate of increase if \$100 placed at interest for 10 years produces \$179.08?

Algebraic Formula.

$$R = \sqrt[n]{\frac{\text{Value at end}}{\text{Value at beginning}}} - 1.$$

Arithmetical Substitution.

$$\text{Rate} = \sqrt[10]{\frac{179.08}{100.00}} - 1.$$

Solution.

$179.08 \div 100 = 1.7908.$
$\text{Log } 1.7908 = 0.253047.$
$\text{Log } 0.253047 \div 10 = \text{Log } 0.025305.$

By reference to the table it is found that log 0.025305 stands for 1.06.

$$1.06 - 1.00 = .06, \text{ or } 6\%.$$

Problems.

1. If the population of a city increases in 5 years from 250,000 to 375,000, find the average ratio of the yearly increase.

2. A machine costing \$900 is found to be worth only \$100 at the end of 10 years. What is the average annual per cent of decrease in value?

Note.—

$$R = 1. - \sqrt[n]{\frac{\text{Value at end}}{\text{Value at beginning}}}$$

3. A building cost \$70,000. The owners at the end of each year deducted 10% from its carrying value as estimated at the beginning of the year. What is the estimated value at the end of 10 years?

$1 = \$70{,}000 \times (.9)^{11-1}.$ Answer, \$24,407.60.

Note.—The value at the end of 10th year is the value at the beginning of the 11th year.

4. In valuing his assets at the close of each year, a manufacturer deducted 10% from the value of his machinery as

listed the previous year, because of depreciation. The machine cost $30,000. What was the value at the end of the 6th year?

5. A building that cost $12,000 has been written down 3% of the decreasing balance each year for 14 years. What is its value as shown by the books?

6. A machine has been depreciated 8% per annum on reducing balance for the past 6 years. Its present value is $450. What was the cost of the machine?

Chapter 22

PARTNERSHIP

Definition.—A partnership results when two or more individuals agree to combine their capitals or services in a business enterprise for the purpose of gain.

Mathematical calculations.—The most important mathematical calculations applicable to partnership accounting are:

1. Division of profits.
2. Liquidation.
3. Calculation of good-will.

Good-will.—The calculation of good-will also applies to the other types of business organization; namely, individual proprietorship and corporation, when changes in ownership, reorganizations, consolidations, and so forth, are made.

Profit-sharing agreements.—Profits may be shared in many ways. A few of the most common methods of profit distribution are:

1. Arbitrary ratios.
2. Ratio of capital invested at organization of business.
3. Ratio of capital accounts at the beginning or at the end of each period.
4. Ratio of average investment.
5. Part of the profits may be distributed as salary or as interest on capital invested, and the remainder in some other ratio.
6. Charge interest on shortage of capital from the agreed investment, and allow interest on excess of capital when such capital is not kept at stated amounts, and distribute the sum or the remainder by agreed ratio.

Lack of agreement.—If the partners have failed to incorporate in their articles of co-partnership an agreement to a

method for distributing profits, the law provides that the profits shall be shared equally, regardless of the ratio of their respective investments.

Losses.—If losses are incurred and no provision has been made for their distribution, the profit-sharing ratio governs.

Illustration.—It seems unnecessary to illustrate the methods of distributing profits equally or by some arbitrary ratio as explained under profit-sharing agreements.

Ratio of investment.—

Example.

January 1, 19—	*A*'s Investment		$10,000
"	*B*'s "		6,000
"	*C*'s "		4,000
	Total		$20,000
December 31, 19—	Profits		$ 4,000

Profits to be shared according to the investment at the beginning of the year.

Solution.

	Investment	*Ratio*		*Profit*	*Shares*
A	$10,000	10/20	of	$4,000 =	$2,000
B	6,000	6/20	of	4,000 =	1,200
C	4,000	4/20	of	4,000 =	800
	$20,000	20/20			$4,000

Explanation.—Add the beginning-of-year investments of each of the partners, take for the numerator of the fraction representing each partner's share his investment at the beginning of the year, and for the denominator the total capital. Using this fraction, calculate the fractional part of the net profit or loss, and this will be the partner's share.

Problems.

Show division of profits in the ratio of investments in the following:

	Investments				
	A	*B*	*C*		
1.	$ 6,000	$ 4,000	$ 2,000	$ 4,000	Net Profit
2.	50,000	20,000	30,000	10,000	" "
3.	1,000	7,500	3,750	2,500	" "
4.	10,000	6,667	3,333	3,000	" "
5.	6,000	10,000	15,000	2,000	Net Loss
6.	3,748	4,365	9,247	3,750	" "

Division of profits by first deducting interest on capital.

Example.

January	1, 19—	*A*'s Investment	$10,000
"		*B*'s "	6,000
"		*C*'s "	4,000
December	31, 19—	Net Profits	4,000

By agreement, profits are to be divided by allowing each partner interest at 5% on his investment (this interest to be deducted from total profits), and the balance is to be distributed equally.

Solution.

A's Investment, $10,000 × .05 = $ 500 Interest
B's " 6,000 × .05 = 300 "
C's " 4,000 × .05 = 200 "
Total ..... = $1000

Net profits, $4,000 — $1,000 = $3,000 to be divided equally.
$3,000 ÷ 3 = $1,000 = each partner's share after interest is deducted.

A:	Interest	$ 500		
	⅓ profits	1000	$1500	Total credit to *A*
B:	Interest	$ 300		
	⅓ profits	1000	$1300	Total credit to *B*.
C:	Interest	$ 200		
	⅓ profits	1000	$1200	Total credit to *C*.
Total			$4000	

Problems.

Show the division of profits in the following:

	Investments			Net Profit	Rate of Int. on Investment	Balance to be Divided
	A	*B*	*C*			
1.	$ 6,000	$ 3,250	$ 2,700	$3,000	5%	Equally
2.	4,000	6,000	9,000	6,000	5%	"
3.	9,250	3,200	12,000	5,000	6%	"
4.	3,725	4,300	4,250	4,300	6%	"
5.	10,000	5,000	5,000	4,000	4%	⅛, ⅜, ⅛
6.	50,000	20,000	10,000	8,000	6%	½, ⅓, ⅙

Profits insufficient to cover interest on investments.— If it is agreed that each partner is to be credited with interest on his investment, the interest must be credited to each part-

ner, even though the total profits are not large enough to cover it. A loss incurred by the distribution of the interest should be divided among the partners in accordance with the agreement as to the division of a profit. The same rule holds good even though there be a loss before interest is credited.

Example.

January	1, 19—	*A*'s Investment	$10,000
"		*B*'s "	6,000
"		*C*'s "	4,000
December	31, 19—	Net Profits	700

By agreement interest is to be allowed each partner at the rate of 5% on his investment and the profits are to be shared equally.

Solution.

A's	Investment,	$10,000 × .05 = $ 500	Interest
B's	"	6,000 × .05 = 300	"
C's	"	4,000 × .05 = 200	"
Total		$1000	

Total interest, $1000 — $700 Profit = $300, deficit to be divided equally.
$300 ÷ 3 = $100, each one's loss.

A:	Interest	$500		
	Less ⅓ loss.....	100	$400	Net credit to *A*.
B:	Interest	$300		
	Less ⅓ loss.....	100	$200	Net credit to *B*.
C:	Interest	$200		
	Less ⅓ loss.....	100	$100	Net credit to *C*.
Total			$700	

Problems.

Find the net credit or debit to each partner in the following:

	Investments			*Net Profit*	*Interest Rate*	*Balance to be Divided*
	A	*B*	*C*			
1.	$10,000	$10,000	$ 5,000	$ 900	6%	Equally
2.	4,000	6,000	2,000	200	6%	"
3.	3,800	4,200	10,000	200	6%	"
4.	9,000	4,000	3,000	375	4%	"
5.	20,000	10,000	5,000	1,200	6%	"
6.	40,000	20,000	10,000	150	4%	"
7.	4,000	3,500	1,500	500	4%	"
8.	3,000	1,000	4,000	4,250	5%	"
9.	20,000	10,000	5,000	6,000	6%	"

Adjustment of capital contribution.—If the partners do not invest the agreed amounts, adjustments may be made, provided the contract so states, by charging each partner with interest on the amount of the shortage of his investment from the agreed amount, and crediting each partner with interest on the excess of his investment over the agreed amount. These adjustments should be made before prorating the profits for the period. If interest adjustments result in a debit balance in the profit and loss account, the net debit balance, or loss, is divided in the ratio of division of profits, unless otherwise agreed.

Example.

		Agreed to Invest	*Invested*
January 1, 19—	*A*	$10,000	$12,000
"	*B*	6,000	5,000
"	*C*	4,000	2,000
December 31, 19—	Profits for the year		$ 3,100

By agreement, *A* is to be allowed 5% interest on his excess investment, and *B* and *C* are to be charged with interest on their shortages. After the above adjustments are made, profits are to be divided equally.

Solution.

A's Excess	$2,000 × .05 =	$100 Interest.
B's Shortage	1,000 × .05 =	50 "
C's Shortage	2,000 × .05 =	100 "

Charge to *B*'s Account.....	$ 50	
Charge to *C*'s Account.....	100	
	$150	
Credit to *A*'s Account.....	100	
Net amount of interest.....	$ 50, to be added to Net Profits.	

Profit before distribution:	Net profit	$3100
	Add interest	50
	Total	$3150

$3150 ÷ 3 = $1050, each partner's share after interest adjustment.

A's ⅓ profits.....	$1050		
Add interest	100	$1150	Total credit of *A*.
B's ⅓ profits.....	$1050		
Less interest	50	$1000	Net credit of *B*.
C's ⅓ Profits.....	$1050		
Less interest	100	$ 950	Net credit of *C*.
Total profits		$3100	

Problems.

1.

	A	*B*	*C*	*D*
Agreed Investment	$5,000	$5,000	$8,000	$2,000
Invested	7,000	6,000	8,000	1,000
Profit ratio after adjustment of 6% Interest on Excess or Deficiency of Capital	40%	20%	30%	10%

Net profit before adjustment, $4,000.

2.

	X	*Y*	*Z*
Agreed Investment	$6,000	$4,200	$4,800
Investment	1,000	4,000	5,000

Net profit for period, $300.

Profit to be shared equally after interest adjustments.

3. The capital of an organization was to be $50,000, of which *A* and *B* were to contribute one-half each, *A* to receive 60% and *B* 40% of profits. *A*, being short of funds, invested only $20,000, and, the firm being short of capital, *B* put in the balance until *A* could make up his shortage, with the provision that he be allowed 5% interest on the excess of his investment over the agreed amount. The profits for the year were $16,000. Show distribution of profits.

Profit sharing in ratio of average investment.—The distribution of profits in the ratio of the average investments may be calculated in two ways.

First method.—Calculate the time each partner's investment was in the business. This may be accomplished by multiplying the original investment by the number of days or months this amount was in the business without change. This product might be termed Day-Dollars, or Month-Dollars.

When the capital is changed, either by additional investment or withdrawal, the changed capital is multiplied by the number of days or months to find its value in day-dollars or month-dollars, and for every change a new calculation is made. The ratio of the total of the day-dollars or month-dollars for each partner to the sum of the day-dollars or month-dollars of all the partners is equal to the ratio of his share of the investment to the total investment.

Example.

	Debit	*A*	*Credit*	
Feb. 1	$1,000		Jan. 1	$10,000
Jun. 1	1,500		May 1	4,000
Nov. 1	500		Jul. 1	1,000
		B		
Jul. 1	$1,000		Jan. 1	$ 6,000
Dec. 1	1,000		Aug. 1	4,000
			Oct. 1	2,000

Net profits of the business for the year, $4,530.

Solution.

Date	*Net Investment*	*Months in Business*	*Month-Dollars*	
		A		
Jan. 1	$10,000	× 1 month	$10,000	
Feb. 1	9,000	× 3 months	27,000	
May 1	13,000	× 1 month	13,000	
Jun. 1	11,500	× 1 month	11,500	
Jul. 1	12,500	× 4 months	50,000	
Nov. 1	12,000	× 2 months	24,000	
A's Month-Dollars Investment				$135,500
		B		
Jan. 1	$ 6,000	× 6 months	$36,000	
Jul. 1	5,000	× 1 month	5,000	
Aug. 1	9,000	× 2 months	18,000	
Oct. 1	11,000	× 2 months	22,000	
Dec. 1	10,000	× 1 month	10,000	
B's Month-Dollars Investment				$ 91,000
Total Month-Dollars Investment				$226,500

A's share of profits $= \frac{135500}{226500}$ of $4530 = $2710

B's share of profits $= \frac{91000}{226500}$ of $4530 = 1820$

$4530

Or if it is desired to find the average investment, it can be found by dividing the month-dollars by 12, as—

A's Month-Dollars, $135,500 ÷ 12 = $11,291.67
B's Month-Dollars, $91,000 ÷ 12 = 7,583.33
Total Average Monthly Investment = $18,875.00

The ratios of the average monthly investments are the same as the ratios of the month-dollars investments.

Second method.—Find the sum of the products of each separate investment; these products are found by multiplying each investment by the number of days or months from the date made until the end of the period. Likewise, find the sum of the products of each withdrawal from the date withdrawn until the end of the period. From the sum of the products of the investments, deduct the sum of the products of the withdrawals, and the result obtained for each partner should be the same as the month-dollars or day-dollars obtained by the first method.

The same problem as above being used, a solution by this method will be given as a check on the previous work.

Solution.

A

Invested — Date	Amount	Time to End of Year	Month-Dollars	
Jan. 1	$10,000 ×	12 months	$120,000	
May 1	4,000 ×	8 months	32,000	
Jul. 1	1,000 ×	6 months	6,000	
				$158,000
— Withdrawals —				
Feb. 1	$ 1,000 ×	11 months	$ 11,000	
Jun. 1	1,500 ×	7 months	10,500	
Nov. 1	500 ×	2 months	1,000	
				22,500
A's Month-Dollars				$135,500

B

Invested				
Jan. 1	$ 6,000 ×	12 months	$ 72,000	
Aug. 1	4,000 ×	5 months	20,000	
Oct. 1	2,000 ×	3 months	6,000	
				$98,000
— Withdrawals —				
Jul. 1	$ 1,000 ×	6 months	$ 6,000	
Dec. 1	1,000 ×	1 month	1,000	
				7,000
B's Month-Dollars				$91,000

The distribution of the profit is the same as in the preceding problem.

Problems.

1.

A

Debit		*Credit*	
Feb. 1	$1,000	Jan. 1	$5,000
Apr. 1	3,000	Mar. 1	3,000
Jun. 1	4,000	Jul. 1	1,000
Aug. 1	500	Oct. 1	2,000
Nov. 1	2,000	Dec. 1	4,000

B

Mar. 1	$4,000	Jan. 1	$6,000
Apr. 1	1,000	May 1	3,000
Dec. 1	2,000	Jul. 1	3,000
		Sep. 1	8,000

The profits for the year were $10,000. Determine the share of each partner, if divided according to average investment.

2.

A

Debit		*Credit*	
Feb. 1	$1,000	Jan. 1	$ 3,000
Mar. 15	200	Aug. 15	4,000
Apr. 1	2,000	Dec. 1	2,000

B

May 1	$2,000	Jan. 1	$10,000
Jun. 1	1,000	Jun. 15	1,000
Sep. 15	500	Oct. 15	1,000

The profits for the year were $7,500. Each partner is to receive interest at 6% on average investment, and the remainder of the profits are to be divided equally. (Use months and half months in calculation.)

3.

A

Debit		*Credit*	
May 1	$2,000	Jan. 1	$15,000
Sep. 15	2,000	Apr. 1	3,000
		Jul. 1	7,000

B

Oct. 15	$5,000	Jan. 1	$20,000
Dec. 1	1,000	May 15	5,000

C

		Jul. 1	$10,000
		Sep. 15	5,000
		Dec. 1	5,000

The profits for the year were $15,000. Show each partner's share, if profits are divided according to average investment.

4. Craig, Brown, and Anderson were partners; after deducting interest on investment and salaries, they were to share the loss or gain in proportion of ⅖ for Craig, ⅖ for Brown, and ⅕ for Anderson. Partners were to be allowed interest at 6% on investment, and charged the same on all money withdrawn. They were also to receive a salary of $2,000 each. Salaries earned to date were credited on July 1. Craig invested, January 1, $12,000, and withdrew, August 1, $4,000. Anderson invested, January 1, $5,600, and withdrew, July 10, $2,100. Brown invested, January 1, $18,000, and withdrew, July 1, $5,500. On the first of the following year, they found they had gained on merchandise, $18,420; on interest (not on partner's accounts) $435.50; on real estate, $1,251.80. The expenses were $9,182.90. What was the present worth of each partner?

5.

A

Debit		*Credit*	
Jan. 25	$200	Jan. 1	$5,000
Apr. 10	250	Feb. 25	1,000
May 3	175	Jun. 10	1,250
Jun. 15	150		

B

Feb. 3	$150	Jan. 1	$5,000
Mar. 15	150	Mar. 10	1,500
May 1	200	May 18	1,275
Jun. 1	100		

The profits for the six months ended June 30 are $1,950. Each partner is to receive 6% interest on his average investment. This interest is to be deducted from the profits. The remainder of the profits is to be distributed, 60% to *A*, 40% to *B*.

Liquidation of partnership.—From the nature of partnerships, they must necessarily be terminated on or before the death of any one of the partners. It is not the purpose to discuss the causes of dissolution, but only the problems met with at the time of settlement. The purpose of the

formation of the partnership is the making of profit, and the division of losses is governed by the same general rule as is the division of profits. Profits should be credited and losses should be charged before making any division of assets. If the above rule is not followed, the result may be an unfair distribution of capital.

When dissolution is accompanied by liquidation, each of the partners has an equal right to share in the work. But as it does not usually require the time of all of the partners, any one of the partners, or an outsider, may wind up the business of the firm.

If, upon complete liquidation, there are profits or losses, they must first be divided in the profit or loss ratio and the remaining capital should be shared by the partners in the capital ratio.

In insolvency, partners must share losses in the profit and loss ratio, and not in the capital ratio. This may at times result in a deficit for some one or more of the partners. Each partner with a deficit should contribute to the firm the amount of his deficit. But if he, too, is totally unable to pay into the firm any portion of his deficit, the remaining partners must bear this loss in their profit and loss ratio.

Methods.—Liquidation may be accomplished in two ways:

1. All the assets converted, all the liabilities paid, the profits or loss distributed, and all of capital divided at one time.

2. A periodic distribution of the capital before all the assets are converted.

Total distribution.—The first method of liquidation does not involve any very difficult calculations.

Example.—From the following figures, show the amount of capital distributed to each partner at dissolution:

	A	*B*	*C*
Investments	$10,000	$6,000	$4,000
Profit Ratio	40%	40%	20%

Assets converted into cash $30,000.
Liabilities to be paid, $14,000.

Solution.

Assets		*Liabilities*		*Net Assets*
$30,000	—	$14,000	=	$16,000

Total investment, $20,000 less Net assets, $16,000 = Loss, $4,000.

	A (40%)	*B (40%)*	*C (20%)*	*Total (100%)*
Investment	$10,000	$6,000	$4,000	$20,000
Distribution of loss...	1,600	1,600	800	4,000
Balances	$ 8,400	$4,400	$3,200	$16,000
Cash distributed	8,400	4,400	3,200	16,000

Periodical distribution.—Periodical distribution may be brought about from two causes:

1. The desire of the partners to reduce the holding of the firm, or completely dissolve it, even though it is still solvent.

2. Forced liquidation.

Usually, in either case, unless the partnership is insolvent, the assets are collected and after the debts are paid the balance is distributed periodically to the partners. If this is done, it should be done in such a way as to reduce the accounts to the profit and loss ratio existing among the partners.

Example.—(Illustrating adjusting capital ratios to profit and loss ratios.) From the following data show the periodic distribution of the cash collected:

	A	*B*	*C*
Investment	$10,000	$6,000	$4,000
Profit ratio	40%	40%	20%
First Period:			
Net loss	$ 1,000		
Cash collected	9,000		
Assets unrealized	10,000		
Second Period:			
Cash collected	$ 5,000		
Net loss	1,000		
Assets unrealized	4,000		
Third Period:			
Cash collected	$ 2,000		

All other assets uncollectible.

Solution.

	A	*B*	*C*	*Total*
Investment	$10,000	$6,000	$4,000	$20,000
Division of loss	400	400	200	1,000
Balance after distribution of loss	9,600	5,600	3,800	19,000
To make a test it will be assumed that the unrealized assets will never be realized.				
Test loss in profit and loss ratio	(4,000)	(4,000)	(2,000)	(10,000)
After deducting the test loss, the remaining amounts will show the proper distribution of the cash balance which should be	5,600	1,600	1,800	9,000
Balance at the end of the first period	4,000	4,000	2,000	10,000
Net loss for period	400	400	200	1,000
Balance after distribution of loss	3,600	3,600	1,800	9,000
The balances of the accounts are now in profit and loss ratio.				
Distribution of cash	2,000	2,000	1,000	5,000
Balance at end of second period	1,600	1,600	800	4,000
Net loss for third period	800	800	400	2,000
Balance after distribution of loss	800	800	400	2,000
Cash distribution	800	800	400	2,000

Example.—(Illustrating adjustments of capital to profit and loss ratio, in which a deficiency of one partner is involved.) Show how each period's cash should be divided in the following:

	A	*B*	*C*
Investment	$10,000	$8,000	$2,000
Profits to be shared equally.			
First Period:			
Net loss			1,500
Cash to be distributed			8,000
Second Period:			
Net loss			1,500
Cash to be distributed			3,000
Third Period:			
All remaining assets sold for			4,000

Solution.

	A	*B*	*C*	*Total*
Invested	$10,000	$8,000	$2,000	$20,000
First Period's loss distributed	500	500	500	1,500
Balance after distributing the actual loss	9,500	7,500	1,500	18,500
Test loss of amount of the unconverted assets	(3,500)	(3,500)	(3,500)	(10,500)

It will be observed by the "Test loss" that *C*'s possible loss is $2,000 greater that his capital. If the test loss should become an actual loss, throwing *C*'s account into a debit balance of $2,000, and if *C* should at that time be unable to pay in this $2,000, *A* and *B* would be required to bear the additional loss. To provide against this contingency a further "test loss" charge of $2,000 is made against *A* and *B*.

Test loss of *C*'s deficiency.....	(1,000)	(1,000)	(2,000)*	
By deducting the sum of the two test losses, $4,500 ($3,500 + $1,000) from *A*'s investment of $9,500, it can be seen that *A* should receive $5,000; it can also be seen that the sum of *B*'s test losses deducted from his investment shows the amount of the cash which is payable to him.				
Cash distribution	5,000	3,000		8,000
Balance of capital undistributed..	4,500	4,500	1,500	10,000
Second Period's losses distributed	500	500	500	1,500
Balance after distribution of actual losses	4,000	4,000	1,000	9,000
Test loss of unrealized assets.....	(2,000)	(2,000)	(2,000)	(6,000)
The same thing is true in this case as formerly. *C*'s account shows a deficit by the test loss. This amount must be distributed as an additional test loss to be taken up by the other partners.				
Test loss for *C*'s deficit.....	(500)	(500)	(1,000)*	
Cash distribution	1,500	1,500		3,000
Balance undistributed	2,500	2,500	1,000	6,000
Third Period's net loss.....	666.67	666.67	666.66	2,000
Cash on hand	1,833.33	1,833.33	333.34	4,000
Cash distributed	1,833.33	1,833.33	333.34	4,000

* Deficiency.

Example.—(Illustrating a return of investments and loans of partners complicated by the accounting principle that "Loans must be paid before capital is returned to partners.")

Partners	*Investment*	*Loan*	*Profit Ratio*
A	$33,000	$10,500	40%
B	28,500	10,000	30%
C	18,000	21,000	20%
D	10,500	18,500	10%
	$90,000	$60,000	100%

The above partners have decided upon a dissolution, and after paying all their liabilities have—

Cash	$ 30,000
Other assets	110,000
Net loss	10,000

How should the cash be divided among partners, assuming that none of the members of the firm has private property?

Solution.

	A	*B*	*C*	*D*	*Total*
Profit and loss ratio...	40%	30%	20%	10%	100%
Investment	$33,000	$28,500	$18,000	$10,500	$90,000
Loss divided	4,000	3,000	2,000	1,000	10,000
Balance of capital undistributed	29,000	25,500	16,000	9,500	80,000
Test loss of assets	(44,000)	(33,000)	(22,000)	(11,000)	(110,000)
Possible deficiency of capital	—(15,000)	—(7,500)	—(6,000)	—(1,500)	—(30,000)

After applying the possible loss of unrealized assets against capital accounts, there is a greater possible loss than the capital invested.

In practical business, it is usually found that as long as the firm has assets and owes each partner money on a loan account, the partner will not pay cash into the firm, as the firm would have to pay it back immediately. The problem states that none of the members of the firm had money other than that invested in the firm. As these shortages are debts due the firm, and as the partners had loan accounts, these loan accounts would undoubtedly be used as a set-off. Therefore, the deficits would be deducted as follows:

	A	*B*	*C*	*D*	*Total*
Loans by partners	$10,500	$10,000	$21,000	$18,500	$60,000
Less possible deficit	—(15,000)	—(7,500)	—(6,000)	—(1,500)	—(30,000)
Financial standing of each partner after distribution of the test loss	—4,500	2,500	15,000	17,000	30,000

As *A*'s loan was not enough to take up the possible deficit, and as the problem stated he had no other property, it is necessary to distribute his test shortage to the other partners.

	A	*B*	*C*	*D*	*Total*
Ratio of distribution....		30/60	20/60	10/60	
Balances after distribution of test loss.....	—$4,500	$2,500	$15,000	$17,000	$30,000
Less *A*'s deficit distributed	(4,500)*	(2,250)	(1,500)	(750)	
Balance after distributing the *A* test deficit..	0	250	13,500	16,250	30,000
This now shows the proper cash distribution		250	13,500	16,250	30,000

* Deficit.

The accounts of the partnership now stand:

Net assets		$110,000	
A Capital			$29,000
B "			25,500
C "			16,000
D "			9,500
A Loan			$10,500
B Loan	$10,000		
Less payment ..	250		9,750
C Loan	$21,000		
Less payment ..	13,500		7,500
D Loan	$18,500		
Less payment ..	16,250		2,250
		$110,000	$110,000

Problems.

1. In the exhibit below, how should the partnership cash be divided? Assume that none of the partners owned other property.

Name	*Capital*	*Loans*	*Profit and Loss Ratio*
A	$11,000	$ 3,000	50%
B	9,500	4,000	30%
C	6,000	8,000	20%
	$26,500	$15,000	

Cash collected ..... $ 5,000.
Other assets ..... 36,500.

2. Show the periodic cash distribution from the following data:

Name	*Capital*	*Loans*	*Profit and Loss Ratio*
A	$10,000	$ 2,000	25%
B	8,000	4,000	25%
C	6,000	6,000	25%
D	4,000	8,000	25%
	$28,000	$20,000	

First Distribution:	
Cash on hand	$ 4,000
Assets	40,000
Loss	4,000
Second Distribution:	
Cash	$30,000
Net loss	2,000
Assets	8,000

3. *A*, *B*, and *C* enter into a partnership agreement to conduct a partnership, and to share profits in the ratio of the investments at the beginning of the year. These investments were, *A*, $6,849.50; *B*, $4,382.45; *C*, $8,246.85. The profits for the year were $3,847.50. State each man's share.

4. The second year they decided on a different profit ratio in which they were to have interest at 6% on the investment at the beginning of the second year, and the balance of the profits were to be shared 40% to *A*, 25% to *B*, and 35% to *C*. The loss for the year was $1,500. Show each man's share.

5. At the beginning of the third year, each man agreed to put in enough cash to make his investment even thousands of dollars. As the firm was short of working capital, *A* invested $2,500, and *B* $1,000 more than the agreed amount, with the provision that they were to receive interest at 5% on excess investment. Partners were to receive credit for the following salaries: *A*, $3,000; *B*, $2,000; *C*, $1,500. After interest adjustments, the remainder of the profits was to be divided equally. The profits were $3,000, before salary or interest adjustments were made.

6. In a partnership, Burr invested $5,492.37, and Wright invested $11,022.85. It was agreed that they were to share the losses and gains in proportion to their investment. At the end of the year the gains and losses were as follows: Real Estate, loss $1,750; horses and wagon, loss $514.69; discount, loss $64.85; merchandise, gain $2,069.58; interest, loss $258.10; merchandise discount, gain $377.75; furniture and fixtures, loss $730.27; expense, loss $1,104.88. They lost on personal accounts, $1,636.60. What is each one's net worth at the end of the year?

7. On the first day of January, *A* and *B* engage in business, each investing $5,000. It is agreed that each one is

to receive interest on his investments, and to be charged interest on his withdrawals. *A* is to receive a salary of $150 per month, and *B* is to receive $100 per month. After the interest and salaries have been deducted, the gains or losses are to be divided equally. On January 20, *A* withdrew $125, and on January 23, *B* withdrew $250. At the end of the month their books showed a loss on discount of $8.61; expense, $482.75. The gains were: interest, $11.03; merchandise discount, $48.75, and merchandise, $1,670.91. Find the net worth of each on February 1st.

8. A partnership agreement states that if any partner fails to invest his agreed share, he shall be charged interest on his shortage at 4½%. The facts are as follows:

	A	*B*	*C*	*D*
Agreed investment	$9,000	$8,000	$7,000	$6,000
Invested	8,000	8,000	8,000	8,000
Profit-sharing ratio	20%	30%	20%	30%

Profits for year, $6,000.

Prepare a statement for the partners, showing the results of your computation.

9.

A				*B*			
19—				19—			
Jul. 1	Invested		$5,000	Jul. 1	Invested		$4,000
Jul. 10	"		2,000	Jul. 6	"		2,000
Jul. 20	Withdrew		1,000	Jul. 20	"		1,000
Aug. 1	"		100	Aug. 1	Withdrew		200
Aug. 15	Invested		300	Nov. 3	"		800
Aug. 31	"		250	Nov. 20	Invested		1,000
Oct. 1	"		500	Nov. 30	"		100
Oct. 24	Withdrew		350	Dec. 2	Withdrew		1,550
Dec. 1	"		200	Dec. 10	"		275

The agreement states that the partners are to share the profits equally after allowing interest at 6% on the average investment. The total profits were $3,450 for the six months. Show the net worth of each partner.

10. The firm of *E. F. & G.* sustained heavy losses, and agree to dissolve. The facts are:

		E	*F*	*G*
Investment, Jan., 1921		$7,000	$3,000	$5,000
First Period:				
Net loss, 1921	$4,000			
Assets uncollected	9,000			
Cash	2,000			
Second Period:				
Assets converted	$7,000			
Losses	2,000			

Show proper distribution of cash for each period.

11. A partnership is to be liquidated by one of the partners, who is to receive 5% of the assets converted as a commission.

	Investment	*Profit and Loss Ratio*
A	$11,200	20%
B	8,400	30%
C	8,000	10%
D	5,000	30%
E	3,000	10%
First Period:		
Net loss		$2,000
Assets converted		6,000
Expenses other than liquidation		500
Liquidation fee		?
Second Period:		
Losses and expenses		400
Liquidation fee		?
Assets converted		8,000
Third Period:		
Losses and expenses		2,500
Liquidation fee		?
The remaining assets all converted		?

Submit a statement showing all transactions in concise form.

12. The capital of three partners, *A*, *B*, and *C*, in a manufacturing business, January 1, 1923, was $26,000, of which *A* owned one-fifth, *B* two-fifths, and *C* two-fifths. On December 31, 1923, one year thereafter, the condition was found to be as follows:

Real estate, $15,000; plant and machinery, $7,000; stock on hand, $2,000; book debts receivable, $6,000; cash in bank, $2,500; creditors' notes payable, $8,000.

Partners' withdrawals,	*A*	(including	interest)	.....	$1,500
"	"	*B*	"	.....	1,200
"	"	*C*	"	.....	2,000

After crediting up interest on capital at the rate of 6%, show the net result for the year, and distribute the same, in proper proportions, to the partners' accounts.

Prepare individual partners' accounts, showing the condition of each at the end of the year.

13. The firm of *C.* & *D.* carry on business as dry goods merchants at Seattle and Tacoma, and decide to dissolve partnership on December 31, 1907. At that date their balance sheet is as follows:

ASSETS

Cash		$ 100.00	
Merchandise—			
At Seattle	$12,000		
At Tacoma	7,250		
		19,250.00	
Accounts Receivable		17,000.00	
Furniture and Fixtures—			
At Seattle	$ 1,700		
At Tacoma	1,100		
		2,800.00	
Good-will		3,000.00	
			$42,150.00

LIABILITIES

Overdraft	$ 1,050.00	
Accounts Payable	11,100.00	
C's Capital	20,000.00	
D's Capital	10,000.00	
		$42,150.00

The partners divide profits in proportion to their respective capital investments, and agree that at the date of dissolution, *C* shall take over the merchandise and furniture at Seattle, and *D* the merchandise and furniture at Tacoma, at the values shown in the above balance sheet. The accounts receivable realize $16,000, and all the liabilities are paid less $300 discount on accounts payable.

The realization is completed by June 30, 1908, and you are required to continue all the accounts appearing in the balance sheet up to and including that date, and to show the final closing of the accounts (including those of the partners). —*C. P. A.—Washington.*

14. Three men, located respectively at New York, Chicago, and Minneapolis, conducted a business as partners.

The partnership agreement recited that the New York partner was to receive a salary of $5,000 a year, the Chicago partner a salary of $3,000 a year and one-third of the profits of that branch, and the Minneapolis partner a salary of $2,000 a year and one-fourth of the profits of that branch. Six per cent interest was to be allowed on the capital used in the business, $500,000, which was employed one-half at New York, one-third at Chicago, and one-sixth at Minneapolis. Repairs and renewals were to be charged to plant, and a depreciation of 6% per annum was to be allowed. Interest on borrowed money was to be distributed according to the invested capital at each branch. Plant account at the beginning of the period showed as follows: New York, $250,000; Chicago, $200,000; Minneapolis, $150,000. Repairs and renewals paid during the year amounted to $25,000 at New York, $30,000 at Chicago, and $15,000 at Minneapolis; interest on borrowed money, $15,000. Subject to these items, the profits of the year amounted to $50,000 at New York, $35,000 at Chicago, and $25,000 at Minneapolis. Complete the profit and loss account for the year.—*C. P. A. —New York.*

15. *A, B, C,* and *D* enter into partnership with a capital of $100,000. *A* invests $40,000, *B* $30,000, *C* $20,000, and *D* $10,000. They are to share profits or losses in the following proportions: *A* 35%, *B* 28%, *C* 22%, and *D* 15%. They are also to receive stipulated salaries chargeable to the business.

At the end of six months there is a loss of $8,000, and meantime the partners have drawn against prospective profits as follows: *A,* $400; *B,* $600; *C,* $600; and *D,* $400.

They dissolve partnership and agree to distribute proceeds of firm assets monthly as realized. *C* and *D* enter other business, and *A* and *B* remain to wind up the firm's affairs, it being stipulated that from all moneys collected and paid over to *C* and *D,* a commission of 5% is to be deducted and divided equally between *A* and *B* for their services in the winding up.

The realization and liquidation lasts four months and the transactions are as follows:

	Assets Realized	*Liabilities Liquidated*	*Expenses and Losses on Realization Exclusive of Commissions*
First month	$ 30,190	$ 7,900	$ 400
Second "	50,300	6,100	750
Third "	20,010	3,800	340
Fourth "	9,500	2,200	110
	$110,000	$20,000	$1,600

Prepare partners' accounts showing the amount payable monthly to each one.—*C. P. A.—New York.*

16. *A, B, C,* and *D* have decided to dissolve partnership. To that end they have liquidated all their liabilities, and at the date of the first division of cash among the partners, the conditions are as follows:

Partners	*Capital*	*Loans*	*Profit and Loss Ratio*
A	$22,000	$ 7,000	40%
B	19,000	6,000	30%
C	12,000	14,000	20%
D	7,000	13,000	10%
Totals	$60,000	$40,000	100%

Cash available for distribution	$ 20,000
Other assets not yet realized (of doubtful value)	80,000
Total	$100,000

State which partners should participate in the distribution of the $20,000; how much cash each should receive; whether the payments should be applied against the capital accounts or the loan accounts. Explain the procedure of determining the distribution. Assume that none of the partners has any private property.—*American Institute Examination.*

17. The capital of a partnership is contributed as follows:

A	$90,000
B	45,000
C	15,000

The partnership agreement provides for profit-sharing as follows:

A	50%
B	30%
C	20%

The partners' salaries are as follows:

A	$5,000
B	3,000
C	2,000

At the end of the first year's business, *C* dies. The books are closed, and the net assets of the business are shown to be $152,500. *A* and *B* liquidate the affairs of the partnership and distribute the surplus as follows:

First distribution	$42,410.20
Second distribution	74,622.30
Final distribution	31,967.50

Prepare a statement of the partners' accounts, showing how the distribution of assets should be made and the apportionment of the loss.—*C. P. A.—Maryland.*

Chapter 23

GOOD-WILL

Definition.—Good-will is an intangible asset, the value of which is based on the capitalized value of the profits of a business which are in excess of a fair return on the net assets.

It can be seen that the value of good-will is the result of forces working together in a going concern. A new establishment should not place a value on good-will, as it will have none, unless the business has been purchased as a going concern.

Valuation.—Two distinct elements enter into the valuation of the good-will of a business:

First, the value of the net assets.
Second, the earning capacity of the concern.

The value of the net assets can best be found by an impartial appraisal which must take into consideration the kind of assets, kind of usage, depreciation, age, obsolescence, and so forth. No mathematical calculation can give a fair value without observing such factors as are mentioned above. Many times the sales price of an establishment is based on the value of the assets only. This is unfair for the seller, and especially so when the value received by the seller is compared with the value of other businesses in which the profits have not been so large in proportion to the net assets.

Methods of valuing good-will.—Three methods of valuing good-will are:

1. Appraisal of good-will.
2. A number of years' purchase price of the net profits.
3. A number of years' purchase price or a capitalization of excess profits over interest on net assets.

There is no particular problem in the calculation of the valuation of good-will by appraisal. It may be appraised by a disinterested party; or more often, it is the amount on

which the vendor and vendee agree. They usually appraise the net assets and agree that the purchase price shall be a certain amount in excess of the value of the assets. This excess is the payment for good-will.

The value of good-will may be estimated to be worth so many years' purchase price of the net or gross profits of any one year or of the average profits of a number of years.

Example.—The consideration of the sale of a business as agreed to between the parties is four years' purchase price of the average profits for the last three years, plus the net value of the net assets.

Net assets		$100,000
Profits 1919	$20,000	
" 1920	15,000	
" 1921	28,000	

What is the selling price of the business?

Solution.

Net assets		$100,000
Profits 1919	$20,000	
" 1920	15,000	
" 1921	28,000	
	$63,000	
$63,000 ÷ 3 = $21,000, Average profits for 3 years.		
$21,000 × 4 = (Good-will)		$ 84,000
Selling price		$184,000

The third method of valuation of good-will may be expanded into two parts:

(*a*) The value may be found by, first, deducting from the average profits a fair return of interest on the capital invested and, second, by multiplying the remainder of the profits, or the excess, by an agreed number of years' purchase price.

(*b*) The value may be found by capitalizing the excess found in (*a*) at an agreed interest rate.

The following is an example of using a number of years' purchase price as a multiplier of the excess profits to find the good-will.

Example.—A agrees to buy a certain business and to pay for it in cash. He agrees to give dollar for dollar of the value of the net assets, plus a six years' purchase price of the excess profits over interest on capital at 6%. Net assets are valued at $100,000, and average profits were $18,000.

Solution.

Net assets		$100,000
Profits average	$18,000	
$100,000 × .06	6,000	
Excess profits	$12,000	
$12,000 × 6 = Good-will		72,000
Purchase price		$172,000

It seems that the fairest value would be determined by using a little higher rate of interest and capitalizing the excess at the same rate of interest, thus:

Net assets		$100,000
Profits average	$18,000	
$100,000 × .08 = Interest	8,000	
Excess	$10,000	
$10,000 ÷ .08 = Good-will		125,000
Purchase price		$225,000

Some objection to this method of calculation is raised because it increases the purchase price so much. This is a very good objection and especially so if the personnel of the management has been the big factor in producing the net profits, and the business passes to new management, but on the other hand it is probably the most fair in consolidations.

Consolidation and capitalization.—Since most phases of the calculation of the value of good-will are found in consolidations, an example of consolidation is given. As it is a very usual custom for a consolidated or reorganized company to issue more than one kind of stock, this feature will be incorporated in the illustration.

Example.—A Company, *B* Company, and *C* Company decide to consolidate under the name of the *X* Company, a corporation. The net assets and average profits of the companies are as follows:

	Net Assets	*Average Net Profits*
A Company	$ 36,000	$ 7,200
B Company	60,000	7,200
C Company	120,000	7,200
Total	$216,000	$21,600

The effect of different methods of consolidation will be shown. Assume that 6% is a fair rate.

The following statement may be prepared from the above stated facts. This statement shows, in addition to the above facts, the amount of interest at 6% on the investment of each company, and also the excess of the average profits over the interest.

	Net Assets	*Average Net Profits*	*6% on Investment*	*Excess Profits*
A Company	$ 36,000	$ 7,200	$ 2,160	$5,040
B Company	60,000	7,200	3,600	3,600
C Company	120,000	7,200	7,200	0
Total	$216,000	$21,600	$12,960	$8,640

By capitalizing the amount of the excess profits, $8,640, at 6%, the good-will will be found to be $144,000.

Method 1.—Distribute the good-will, $144,000, in proportion to the assets, giving common stock for the good-will, and 6% preferred stock for the assets.

	6% Preferred Stock	*Common Stock*	*Total Stock*
A Company	$ 36,000	$ 24,000	$ 60,000
B Company	60,000	40,000	100,000
C Company	120,000	80,000	200,000
	$216,000	$144,000	$360,000

Method 2.—Distribute the good-will equally among the companies. This distribution is equivalent to a capitalization of the average net profits on a 6⅔ years' basis, giving 6% preferred stock for the assets.

	6% Preferred Stock	*Common Stock*	*Total Stock*
A Company	$ 36,000	$ 48,000	$ 84,000
B Company	60,000	48,000	108,000
C Company	120,000	48,000	168,000
	$216,000	$144,000	$360,000

Method 3.—Distribute the good-will on a basis of capitalization of the excess profits of each company at 6%, giving 6% preferred stock for the net assets.

The amounts of common stock distributed for good-will are calculated by dividing the excess profits of each company by the rate per cent. The excess profits of the *A* Company amount to $5,040. If this is divided by .06, the quotient will be $84,000. The same calculation is made for the *B* Company. As the *C* Company has no excess profits, it will not receive any common stock. The results of these calculations are shown in the following statement:

	6% Preferred Stock	*Common Stock*	*Total Stock*
A Company	$ 36,000	$ 84,000	$120,000
B Company	60,000	60,000	120,000
C Company	120,000		120,000
	$216,000	$144,000	$360,000

A comparison of results may be beneficial. Assume that the profits have been neither increased nor decreased by the consolidation, that all the profits were distributed, and that the stock given for the asset is 6% preferred, non-participating.

A statement of the distribution of profits for comparative purposes might be as follows:

Method 1.—

DISTRIBUTION OF DIVIDENDS

	6% Both Preferred and Common	*After Consolidation*	*Before Consolidation*	*Increase or Decrease* *
A Preferred Stock	$2,160			
Common Stock	1,440	$ 3,600	$ 7,200	*$3,600
B Preferred Stock	3,600			
Common Stock	2,400	6,000	7,200	* 1,200
C Preferred Stock	7,200			
Common Stock	4,800	12,000	7,200	4,800
Total		$21,600	$21,600	

C Company will surely favor this method of distribution of profits, but the stockholders of *A* Company and *B* Company will not receive as large a dividend in proportion to their investment as before.

Method 2.—

DISTRIBUTION OF DIVIDENDS

	6% Preferred and Common Stock	*After Consolidation*	*Before Consolidation*	*Increase or Decrease **
A Preferred Stock	$2,160			
Common Stock	2,880	$ 5,040	$ 7,200	* $2,160
B Preferred Stock	3,600			
Common Stock	2,880	6,480	7,200	* 720
C Preferred Stock	7,200			
Common Stock	2,880	10,080	7,200	2,880
Total		$21,600	$21,600	

While this distribution of profits shows better results than the former, it is still unjust to *A* Company and *B* Company.

Method 3.—

DISTRIBUTION OF DIVIDENDS

	6% Preferred and Common Stock	*After Consolidation*	*Before Consolidation*	*Increase or Decrease **
A Preferred Stock	$2,160			
Common Stock	5,040	$ 7,200	$ 7,200	None
B Preferred Stock	3,600			
Common Stock	2,600	7,200	7,200	None
C Preferred Stock	7,200	7,200	7,200	None
Total		$21,600	$21,600	

With increased profits.—While it seems from the above that the last method is the only fair and just one, other elements might enter to change our conclusions. Assume that, because of the consolidation, the profits were increased 10%. The profits would be $23,760.

If the preferred stock were 6% participating, and the profits were distributed in dividends, the following would be the result:

The preferred stock would first receive a 6% dividend; the common stock would receive a 6% dividend; and the balance of the profits, $2,160, would be divided between the common and preferred stock in proportion to the total shares of each class of stock issued. This would result in an extra dividend of .6%.

Method 1.—

	6% Dividend Preferred Stock	6% Dividend Common Stock	.6% Dividend All Stocks	Total Dividend
A Preferred Stock	$ 2,160		$ 216	
Common Stock		$1,440	144	$ 3,960
B Preferred Stock	3,600		360	
Common Stock		2,400	240	6,600
C Preferred Stock	7,200		720	
Common Stock		4,800	480	13,200
	$12,960	$8,640	$2,160	$23,760

Method 2.—

	6% Dividend Preferred Stock	6% Dividend Common Stock	.6% Dividend All Stocks	Total Dividend
A Preferred Stock	$ 2,160		$ 216	
Common Stock		$2,880	288	$ 5,544
B Preferred Stock	3,600		360	
Common Stock		2,880	288	7,128
C Preferred Stock	7,200		720	
Common Stock		2,880	288	11,088
	$12,960	$8,640	$2,160	$23,760

Method 3.—

	6% Dividend Preferred Stock	6% Dividend Common Stock	.6% Dividend All Stocks	Total Dividend
A Preferred Stock	$ 2,160		$ 216	
Common Stock		$5,040	504	$ 7,920
B Preferred Stock	3,600		360	
Common Stock		3,600	360	7,920
C Preferred Stock	7,200	0	720	7,920
	$12,960	$8,640	$2,160	$23,760

Assume that because of the consolidation the net profits were increased 10%, and that the stock distribution is the same as in the previous example, except that the preferred stock is 6% preferred, non-participating. The preferred stockholders would receive 6% on their stock. If the remaining profits were distributed to the common stockholders, they would receive a dividend of 7½% on their holdings.

Method 1.—

	6% Dividend Preferred Stock	7½% Dividend Common Stock	Total Dividend
A Preferred Stock	$ 2,160		
Common Stock		$ 1,800	$ 3,960
B Preferred Stock	3,600		
Common Stock		3,000	6,600
C Preferred Stock	7,200		
Common Stock		6,000	13,200
	$12,960	$10,800	$23,760

Method 2.—

	6% Dividend Preferred Stock	7½% Dividend Common Stock	Total Dividend
A Preferred Stock	$ 2,160		
Common Stock		$ 3,600	$ 5,760
B Preferred Stock	3,600		
Common Stock		3,600	7,200
C Preferred Stock	7,200		
Common Stock		3,600	10,800
	$12,960	$10,800	$23,760

Method 3.—

	6% Dividend Preferred Stock	7½% Dividend Common Stock	Total Dividend
A Preferred Stock	$ 2,160		
Common Stock		$ 6,300	$ 8,460
B Preferred Stock	3,600		
Common Stock		4,500	8,100
C Preferred Stock	7,200		7,200
	$12,960	$10,800	$23,760

It can be seen that in case of increased earnings, Method 3 is extremely unfavorable to Company *C*.

No method of capitalization can be said to be perfect, and, after all, the above are customary methods and will serve only as guides. Circumstances must govern all agreements.

Problems.

1. Suppose the following companies desire to combine. The good-will is to be adjusted so that it will be fair to each company.

	A	*B*	*C*	*Total*
Net assets	$200,000	$300,000	$500,000	$1,000,000
Average profits	40,000	45,000	65,000	150,000
Interest rate				7½%
Profit rate	20%	15%	13%	15%

Show by means of tables, methods of capitalization similar to those illustrated.

2. With the above information, assume that the profits would be increased 10% by the consolidation. Show by comparative tables, the results of the different methods of capitalization.

3. The corporations *A*, *B*, and *C*, call upon you to draw up financial plans for their consolidation. They submit the following information:

Assets	*A*	*B*	*C*
Plants	$300,000	$180,000	$ 80,000
Materials	100,000	20,000	20,000
Accounts receivable	80,000	60,000	40,000
Cash	20,000	10,000	10,000
Liabilities			
Accounts payable	$ 70,000	$ 30,000	$ 20,000
Capital	350,000	200,000	100,000
Surplus	80,000	40,000	30,000
Average income	32,000	36,000	48,000

Assume a fair rate of return on a business of this character to be 8%. Draw up a memorandum of consolidation showing the kinds of securities that are to be issued by the consolidated company, the distribution of the securities and the increase in income to the stockholders of each company.

(*a*) Divide the stock partly on the basis of good-will and partly on the basis of net assets.

(*b*) Base the distribution entirely on earning power.

(*c*) Assume the company will be able to earn $20,000 more than the total earnings of the separate companies. This increased earning will be due to good management.

(*d*) Assume that the $20,000 increased earnings will be due to the consolidation of assets.

4. John Doe in Milwaukee, the *A* Company in Green Bay, and the *B* Company in Appleton are engaged in the same line of business.

The proposition has been made for a corporation (*C* Company) to take over the business of these three concerns, and the owner and stockholders have tentatively agreed to the plan.

You have been called in to lay before the parties four plans or bases of consolidation and resulting capitalization; one plan upon the assumption that the earning capacity will not increase through the consolidation, and the other plans upon the assumption that the earnings will increase 10%.

The assets, liabilities, and net worth of the concerns are given herewith. The inventory and valuation figures are those certified by the appraisal company and are accepted by all parties concerned. Each of the concerns has agreed to consider the assets and liabilities stated as the average assets and liabilities employed in the business.

An average income statement covering a period of five years is given herewith, and should receive due consideration in working out plans of consolidation and division of capital stock. Care should be given to your arrangement of the figures so that comparisons may be easily made.

ASSETS, LIABILITIES, AND NET WORTH

Assets	*John Doe*	*A Co.*	*B Co.*	*Total*
Cash	$ 3,000	$ 1,500	$ 10,000	$ 14,500
Notes receivable	2,000	5,000	3,000	10,000
Accounts receivable	7,000	20,500	30,000	57,500
Materials and supplies	15,000	30,000	70,000	115,000
Equipment	20,000	25,000	46,000	91,000
Office furniture	2,000	3,000	5,000	10,000
Real estate	4,000	6,000	15,000	25,000
Buildings	10,000	24,000	40,000	74,000
Totals	$63,000	$115,000	$219,000	$397,000
Liabilities and Net Worth				
Notes payable	$10,000	$ 7,000	$ 4,000	$ 21,000
Accounts payable	8,000	18,000	35,000	61,000
John Doe's investment	45,000			45,000
Capital stock		75,000	160,000	235,000
Surplus		15,000	20,000	35,000
Totals	$63,000	$115,000	$219,000	$397,000

AVERAGE INCOME STATEMENT

Sales	$200,000	$350,000	$600,000	$1,150,000
All operating expenditures	187,400	337,400	587,400	1,112,200
Net income from operation	$ 12,600	$ 12,600	$ 12,600	$ 37,800
Non-operating revenue	$ 1,000	$ 13,600	$ 39,800	$ 54,400
Non-operating expenditures	3,000	5,000	10,000	18,000
	$ 10,600	$ 21,200	$ 42,400	$ 74,200

(Use 7% in this problem.)

—*C. P. A.—Wisconsin.*

5. Corporations *L, M, N, O* and *P* decide that it is to their mutual advantage to amalgamate.

The following information is on file regarding these corporations:

	Present Capital Stock and Surplus	*Average Annual Net Earnings 10-year Period*	*Appraised Value, Net Assets*
L	$ 5,250,000	$ 420,000	$ 4,000,000
M	2,500,000	300,000	3,000,000
N	750,000	150,000	1,200,000
O	500,000	110,000	700,000
P	1,000,000	180,000	1,500,000
	$10,000,000	$1,160,000	$10,400,000

The directors of the *L* corporation suggest that the capitalization of the new company be $16,000,000 and that each corporation receive its pro rata share in accordance with the present capitalization, *L* receiving $8,400,000, *M* $4,000,000, *N* $1,200,000, *O* $800,000, and *P* $1,600,000.

The plan is favored by all the companies, subject to the approval of their accountants.

On behalf of the *M, N, O,* and *P* corporations, you are asked to criticize the plan and report, submitting an alternate plan if you consider the one proposed is not equitable to any or all of the companies which you are representing.—*American Institute Examination.*

Chapter 24

FINANCING

Value of stock "rights."—A stock "right" is the privilege of a stockholder to subscribe to one or more shares of stock in a corporation at a price less than the market price. This, of course, has reference to stock selling above par. The term "right" on some stock exchanges indicates the privilege of subscribing to one share of the new issue, while the term "right" on the New York Exchange indicates the privilege of the owner of each share of outstanding stock to subscribe to a share or part of a share in the new issue.

Example.—A corporation, whose capital is $100,000, has decided to increase its stock to $150,000 (par value of each share $100). The market value of each share is $150. It decides to give the present holders the privilege of buying this new stock at $125. What is the value of the "right," if stockholders are privileged to subscribe to one-half their present holdings?

Formula.	*Substitution.*
$\frac{\text{Market price} - \text{Offering price}}{\text{Number of rights to purchase 1 share, } + 1} = \text{Value of rights.}$	$\frac{150 - 125}{2 + 1} = \frac{25}{3}$ or $8.33.

Solution.

Market value of 1000 shares at 150	= $150,000
Proposed selling price of 500 shares at $125 to stockholders	= 62,500
	$212,500

Dividing the total capitalization by the number of shares, gives the value of one share; thus,

$212,500 ÷ 1500 = $141.66.
$ 141.66 — 125 = $ 16.67, value of 2 "rights."
$ 16.67 ÷ 2 = $ 8.33, value of each "right."

The value of stock rights to an individual for Income Tax purposes is calculated on the cost of the stock rather than on the market price, as given in the above formula.

For income tax purposes.—To the cost of the share or shares required to purchase one new share, add the subscription price of the new share and divide the sum by the total number of shares; the result will be the average price of the share. Subtract the average cost from the market price to obtain the profit on the sale of the right.

Example.—Find the value of the stock rights for Income Tax purposes, if *A* paid $140 each for two shares of stock.

Solution.

Cost of 2 shares of stock at $140 each	$280
Subscription price of one new share	125
Total cost of three shares	$405
Cost of one share (⅓ of $405)	$135
Market value	150
Less cost of shares	135
Profit on sale of stock right for tax purposes	$ 15

Problems.

1. Suppose a corporation has a million dollars par value of stock outstanding, with a market value of $150 per share, and determines to issue $500,000 more of the same class of stock, with a subscription privilege to the present shareholders of buying the new stock at $125. What, under these conditions, is the value of the "right" accorded to each share of the old stock to purchase one-half a share of the new stock at the favorable price per share of $125?

2. Assume that *A* owns 80 shares of stock in *X* Corporation. The *X* Corporation has an outstanding capital of 10,000 shares (par value $100) of common stock, and the directors decide to increase the outstanding stock to 12,500 shares, offering all of the new stock to the stockholders as a special privilege at $120 per share. If the market value is $140 per share, what is the value of the right to *A*?

3. Calculate the value of *A*'s stock rights for Tax purposes, if *A* paid $115 per share for his stock.

Working capital.—One of the most difficult problems for a man entering a new business, is to know how much money will be required to finance the enterprise until the receipts will

equal or exceed the disbursements. While this is strictly a question of finance, it often confronts the accountant.

Assume that a manufacturer gives you the following facts and requests you to give him an estimate of the amount of working capital required to finance the making and selling of an article:

Selling price, each	$100	
Cost to make, each	60	
Selling expenses, each	20	
Overhead, each	10	
Net profit, each	10	
Sales, first month	50	articles
" second "	100	"
" third "	150	"
" fourth "	200	"
" each month thereafter	200	"

Sales are all made on installment payments of $10 each month. Arbitrarily assume that the complete cost of $90 on each article is incurred at the time the sale is made.

What will be the largest amount of capital required and in which month will it occur?

Solution.

	Total Costs Each Month	*Total Receipts Each Month*	*Deficiency Each Month*	*Working Capital Required*
First month	$ 4,500	$ 500	$ 4,000	$ 4,000
Second "	9,000	1,500	7,500	11,500
Third "	13,500	3,000	10,500	22,000
Fourth "	18,000	5,000	13,000	35,000
Fifth "	18,000	7,000	11,000	46,000
Sixth "	18,000	9,000	9,000	55,000
Seventh "	18,000	11,000	7,000	62,000
Eighth "	18,000	13,000	5,000	67,000
Ninth "	18,000	15,000	3,000	70,000
Tenth "	18,000	17,000	1,000	71,000
Eleventh "	18,000	18,500	500 *	70,500
Twelfth "	18,000	19,500	1,500 *	69,000

* Receipts from collections are more than the costs for the month.

The above table shows in the last column the amount of working capital required to finance the business by months. The largest amount required is found to be $71,000 in the tenth month. The collections thereafter are larger than the costs.

Problems.

1. A corporation is about to engage in the manufacture and sale of the "Porto" Typewriter. It has made the following estimates after careful investigation, and desires to know the approximate working capital required.

REPORT ON "PORTO" TYPEWRITER

Selling price, each	$65
Cost of make, each	30
Selling expense, each	10
Collection expense and loss:	
$1 a month for 10 months	10
Net profit	15

Terms of payment: $15 down, $5 a month.

Sales:		
First month	40	typewriters
Second "	80	"
Third "	130	"
Fourth "	190	"
Each month thereafter	200	"

2. You are called in by a Typewriter Company, selling upon the installment plan, to compute the amount of working capital to be required for the year 1924.

Your examination discloses the following facts: The retail price of the machine is $100; the cost of material is 40% of cost of sales; and the direct labor is 35% of the cost of sales. Burden does not include selling, administration, or collection expense and loss. Selling expense is 12% of sales, administration expense is 5% of sales; collection expense and loss is an additional 3% of sales. You also find that future labor cost will increase 14⅔%, which will decrease the profit 35%.

Terms of sales are: $10 with order and balance in nine equal installments. Estimated sales, first month fifty machines, with an increase of 50 machines sold each month up to fourth month, when 200 machines were sold, with no further increase during year.

Assume all expense must be paid during month in which sale is made. Prepare schedule showing all essential facts, and amount of working capital needed monthly.—*C. P. A.—Wisconsin.*

Cumulative voting.—Cumulative voting is a method whereby each shareholder may cast as many votes as he holds shares multiplied by the number of directors to be elected. He may cast all of his votes for any one or more of the directors to be elected, or distribute his votes in any way he may desire. In this way, the minority stockholders may combine their votes in such a way that they may be sure of electing a representative on the board of directors.

Example.—A corporation has an outstanding capital stock of $100,000, composed of 1000 shares of common stock with a par value of $100 each. The stockholders are to elect seven directors at the annual meeting. Calculate the least number of shares required to elect three of the directors, provided that cumulative voting is used.

Formula.

$$\frac{a \times c}{b + 1} + 1 = x.$$

Substitution.

$$\frac{1000 \times 3}{7 + 1} + 1 = 376.$$

Explanation.—

a = Shares outstanding.
b = Directors to be elected.
c = Directors desired.
x = Required number of shares.

Verification.

$1000 \times 7 = 7000$, total votes of 1000 shares.
$376 \times 7 = 2632$, " " " 376 "
$2632 \div 3 = 877$, the number of votes each of the three directors would receive if the owners of the 376 shares of stock cast all their votes for three directors.
$7000 - 2632 = 4368$, balance of votes.
$4368 \div 5 = 873$, the largest number of votes the remaining stockholders could cast for five directors.

By the same method of calculation, it will be found that the owners of 375 shares would have 875 votes for each of three directors, while the remaining stockholders would have 875 votes for each of five directors. This would give a tie vote, and neither side could elect the desired number.

Problems.

1. In a corporation which has cumulative voting, how many of the five directors can you safely seek to elect, if you own 501 out of the 3000 voting shares?

2. Assume that nine directors are to be elected by the *X* Company, which has a voting capital stock of 5000 shares. How many shares are necessary to elect six directors under the cumulative plan?

Book value of shares of stock.—A corporation is owned by the stockholders, whose evidences of ownership are shares of stock. The fact that a share of stock has $100 printed on its face does not necessarily mean that it is worth that amount. It may be worth more than $100 a share, or it may be worth less than that. The amount printed on the face of a certificate of stock is its *par value.* The *book value* of a share of stock is equal to the quotient of the net assets divided by the number of shares of stock.

Formula.—

$$\frac{\text{Assets} - \text{Liabilities}}{\text{Number of shares}} = \text{Book value per share.}$$

Example.—A corporation has assets of $340,000, and liabilities of $120,000. It has a capital stock of $200,000. If the shares have a par value of $100 each, what is their book value?

Solution.

$$\$340{,}000 - \$120{,}000 = \$220{,}000.$$
$$\$220{,}000 \div 2000 \text{ shares} = \$110.$$

Problems.

1. What is the book value in each of the following:

	Assets	*Liabilities*	*Capital*	*Par Value of Shares*
(*a*)	$ 1,322,073.35	$ 105,804.19	$ 1,000,000.00	$100.00 each
(*b*)	81,187,057.34	50,655,296.13	23,989,300.00	$100.00 each
(*c*)	349,328.55	80,784.13	50,000.00	$ 50.00 each
(*d*)	368,112.51	126,517.80	150,000.00	$ 50.00 each

2. An eastern company has assets of $376,915.49, and liabilities $157,907.71. There are 5000 shares outstanding, with par value $50. The balance sheet shows a deficit of $30,992.22. What is the book value of each share?

Chapter 25

INSTALLMENT SALES

Installment sales.—"Dealers in personal property ordinarily sell either for cash, or on the personal credit of the buyer, or on the installment plan. Dealers in personal property who sell on the installment plan may treat the obligations of purchasers as the equivalent of cash, and take up the profit in the period the contract is made, or they may take up as income that proportion of each installment payment which the gross profit, to be realized when the property is paid for, bears to the gross contract price.

"Such income may be ascertained by taking as profit that proportion of the total cash collections received in the taxable year from installment sales (such collections being allocated to the year against the sales of which they apply), which the annual gross profit to be realized on the total installment sales made during each year bears to the gross contract price of all such sales made during that respective year." Article 42, of Regulations 62.

Example.—The books of the *X Y Z* Company, selling merchandise on the installment plan, show the following:

	1918	*1919*	*1920*	*1921*
Sales	$ 89,257.99	$111,825.86	$137,012.32	$ 97,912.26
Cost of Sales:				
Inventory (Old) ...	$ 45,030.85	$ 43,369.05	$ 53,213.44	$ 48,068.06
Purchases	57,633.30	73,601.88	93,738.31	53,543.85
	$102,664.15	$116,970.93	$146,951.75	$101,611.91
Less: Invt. (New)..	43,369.05	53,213.44	48,068.06	42,868.36
Cost of Sales.....	$ 59,295.10	$ 63,757.49	$ 98,883.69	$ 58,743.55
Gross Profit	$ 29,962.89	$ 48,068.37	$ 38,128.63	$ 39,168.71

In 1921, cash was collected on the contracts as follows:

1918	*1919*	*1920*	*1921*
$1,635.35	$4,832.00	$25,182.14	$69,927.92

(*a*) What is the income for 1921, and (*b*) what amount should be credited to "Reserve for Unearned Gross Profits" for 1921?

Solution.

(*a*) PER CENT OF GROSS PROFIT

1918 $29,962.89, Gross profit ÷ $ 89,257.99, Sales = 33.57%.
1919 $48,068.37, Gross profit ÷ $111,825.86, Sales = 42.98%.
1920 $38,128.63, Gross profit ÷ $137,012.32, Sales = 27.83%.
1921 $39,168.71, Gross profit ÷ $ 97,912.26, Sales = 40.00%.

PROFIT ON COLLECTIONS

1918	$ 1,635.35, Amount collected × 33.57% =	$ 548.99
1919	$ 4,832.00, Amount collected × 42.98% =	2,076.79
1920	$25,182.14, Amount collected × 27.83% =	7,008.19
1921	$69,927.92, Amount collected × 40.00% =	27,971.17
	Gross profit realized in 1921 =	$37,605.14

(*b*) $39,168.71, Gross profit on sales for 1921, — $27,971.17, the Gross profit realized on sales for 1921, = $11,197.54 to be credited to "Reserve for Unearned Gross Profits."

Bad debts.—The bad debts written off during the year should be allocated by years, and the charge should be made to "Reserve for Unearned Gross Profits" for the percentage of gross profit in each year's write off, and to Profit and Loss (Bad Debts) for the remainder, the entire credit being made to Accounts Receivable.

Example.—In 1921, bad accounts were written off as follows:

1918	*1919*	*1920*	*1921*
$67.65	$141.05	$65.62	$126.25

(*a*) What amount should be charged against "Reserve for Unearned Gross Profits" and (*b*) what amount should be charged to Profit and Loss?

Solution.

		Unrealized Profit	*Balance*
1918	$ 67.65, Amount written off × 33.57% =	$ 22.71	$ 44.94
1919	$141.05 " " " × 42.98% =	60.62	80.43
1920	$ 65.62 " " " × 27.83% =	18.26	47.36
1921	$126.25 " " " × 40.00% =	50.50	75.75
	$400.57	$152.09	$248.48

Entry:

Reserve for Unearned Gross Profit....	$152.09	
Profit and Loss (Bad Debts).....	248.48	
To Accounts Receivable		$400.57

Problems.

1. The *X Y Z* Company's books for 1922 showed the following:

Sales	$120,183.19
Cost of Sales:	
Inventory (Old)	$ 42,868.36
Purchases	82,690.89
	125,559.25
Less Inventory (New)	45,256.20
Cost of Sales	80,303.05
Gross Profit	$ 39,880.14

Collections were made in 1922 as follows:

1918	*1919*	*1920*	*1921*	*1922*
$219.25	$1,219.14	$8,472.25	$19,701.90	$77,522.27

Calculate (*a*) the income for 1922, and (*b*) the amount to carry to "Reserve for Unearned Gross Profits." Use the rates given in the solution for the years prior to 1922.

2. The analysis of bad debts in 1922 was as follows:

1918	*1919*	*1920*	*1921*	*1922*
$8.35	$209.77	$840.07	$168.92	$325.65

What amounts should be charged to "Reserve for Unearned Gross Profits" and "Profit and Loss," respectively?

3. Results for 1923:

Sales	$122,385.49
Cost of Sales:	
Inventory (Old)	45,248.20
Purchases	82,495.38
	127,743.58
Less: Inventory (New)	46,498.75
Cost of Goods Sold	81,244.83
Gross Profit	41,140.66

Collections were:

1918	*1919*	*1920*	*1921*	*1922*	*1923*
$55.85	$437.62	$2,561.52	$4,338.12	$25,465.67	$79,915.86

(*a*) Calculate per cent of gross profit for 1923.
(*b*) Calculate income for 1923.

(*c*) Calculate amount of credit to "Reserve for Unearned Gross Profits."

4. Bad debts analysis for 1923 was as follows:

1919	*1920*	*1921*	*1922*	*1923*
$58.73	$28.26	$43.12	$144.39	$111.76

Calculate the charges to "Reserve for Unearned Gross Profits," and to "Profit and Loss."

Chapter 26

FOREIGN EXCHANGE

Foreign trade.—The foreign trade of American concerns has grown so extensively that it is necessary for the accountant to be acquainted with the basic principles of exchange values.

Rate of Exchange.—Theoretically, the rate of exchange between any two countries ought to be the ratio between the values of the amounts of metal in their standard monetary units. This theoretical rate is commonly known as "par of exchange," but because of the many economic factors entering into foreign business, another rate is used, called the "current rate."

Par of exchange.—Theoretically, the rate of exchange between any two countries ought to be the ratio between the amounts of gold, if the countries are on a gold basis, or silver, if on a silver basis, contained in their standard monetary units. If one country is on a gold basis and the other is on a silver basis, it is necessary to determine the number of grains of pure metal in the standard coin of each country. By using the grains of pure metal in connection with the value of one metal in terms of another, the ratio or par of exchange may be found.

Example.—Find the par of exchange of the English sovereign and the United States gold dollar.

Solution.

	Grains
English sovereign weighs.....	123.27447
Less alloy 8⅓%	10.2728725
Net weight of gold.....	113.0015975
United States gold dollar.....	25.8
Less alloy 10%.....	2.58
Net weight of gold.....	23.22

113.0015975 ÷ 23.22 = 4.866563 +, or par rate.

A comparative table showing the value of foreign coins is issued at frequent intervals by the United States Treasury Department.

Current rate of exchange.—The current rate of exchange may, under normal conditions, be approximately at par, but at the present time the current rate of exchange is in many cases quite different from par. This is due to many causes, some of which are:

Shortage of gold in foreign countries.

Issuance of large amount of paper money by foreign countries.

Abnormal trade balances.

Lack of confidence in foreign governments.

Time element entering into the date of settlement.

Quotations are usually made on "demand," but may be made on thirty, sixty, or ninety day drafts. In calculations, where time is involved, three days of grace are ordinarily allowed.

The following was taken from one of the financial magazines, and illustrates the methods of quotations:

THE RANGE OF DEMAND EXCHANGE FOR THE WEEK ENDING OCTOBER 20, 1922

	High	*Low*	*Last*	*Net Change*	*Parity*
London, stlg.	$4.48 1/16	$4.43 3/8	$4.46 3/8	.03 1/8	$4.8665
Paris, fr.	.0758	.0737	.0737	—.0020 1/2	.193
Italy, lr.	.0424 1/2	.0418	.0418 1/2	—.0006	.193
Belgium, fr.	.0705 1/2	.0683	.0683	—.0023	.193
Switzerland, fr.	.1852	.1826	.1826	—.0031	.193
Germany, mk.	.0003 11/16	.0003 1/2	.0002 3/4	—.0001 1/16	.2382
Holland, flr.	.3918	.3888	.3911	.0022	.402
Spain, psat.	.1542	.1525	.1535	.0009	.193
Greece, drch.	.0268	.021	.0212	—.0056	.193
Sweden, krn.	.2673	.2665	.267	.0003	.268
Norway, krn.	.182	.1765	.1765	—.0068	.268
Denmark, krn.	.2008	.1991	.1993	—.0023	.268
Japan, yn.	.4813	.48	.4806	.0006	.4984
Argentina, ps.	.3633	.359	.363	.004	.4244
Brazil, mrs.	.1165	.1130	.113	—.0035	.3245
Canada	1.00 9/64	1.00 3/32	1.00 9/64	.00 3/64	1.00

Six classes of problems.—The mathematics of foreign exchange may be resolved into six classes of problems:

1. Conversion of one monetary unit into terms of another.

2. Interest on foreign exchange.
3. Bankers' and brokers' problems of valuing time bills of exchange.
4. Finding the value of an account as a whole.
5. Averaging accounts of foreign exchange with interest.
6. Foreign branch house accounting.

Conversion of one monetary unit into terms of another.—The first class of problems results from the necessity of converting one monetary unit into terms of another as a result of the simple purchase of letters of credit, travelers' checks, and cable transfers on a foreign country. These transfers are made without reference to time and are calculated at the quoted or current rate of exchange. This current rate is the rate prevailing for that class of transfers on the date of purchase.

Reduce the foreign exchange to the basis of the conversion rate, multiply by the conversion rate, and add the charges for the transfer.

Example 1.—John Smith desired to purchase a letter of credit on London for £150 7s. 6d. (The bank charge in this case is ½% for its part of the transaction.)

Use the rate of exchange quoted in the table on page 190 ($4.46⅜), and find the total cost.

Solution.

	£150
7s. = 7/20 of a £	.35
6d. = 6/240 of a £	.025
	£150.375
Multiply by rate of exchange	4.46⅜
Total cost in dollars	$671.24
Bank charges usually ½%	3.36
Total cost	$674.60

Sometimes it is desired to know how much foreign exchange can be purchased for a definite number of dollars.

Example 2.—*A* has $1000 which he desires to send to his family in France. How many francs can he buy? The rate of exchange is quoted at $.0737.

Solution.

If 1 franc costs \$.0737, \$1000 will buy as many francs as \$.0737 is contained in \$1000, or 13,568+ francs.

Divide the number of dollars by the conversion rate of exchange. The quotient will be the value on the converted basis. If this quotient or a part of it is a decimal, this decimal must be reduced to the next lower order of converted money.

Example 3.—What is the value in London exchange of a draft that cost \$1000 in United States money? The rate quoted is \$4.46⅜.

Solution.

£1000 ÷ \$4.46375 = 224.0269, or number of pounds sterling.
.0269 × 20 = .538s.
.538s. × 12 = 6.456d.

Answer: £224 6d.

Problems.

At the rates quoted on page 190, find the cost in United States money of:

1. £1200 16s.
2. £8240 5s. 8d.
3. 10000 francs.
4. 92500 marks. Commission ¼%.
5. £1246 4s. 3d. Commission 1%.
6. 7240 francs. Commission ½%.

How much exchange at the rates quoted previously can be purchased for:

7. \$1000 on England.
8. 250 on Japan.
9. 7550 on Brazil.
10. 1240 on Holland.
11. \$ 250 on Italy.
12. 246 on London.
13. 8962 on Canada.
14. 3200 on Belgium.

Interest on foreign exchange.—To find the interest on foreign exchange:

1. Reduce the sum of the exchange to terms of one monetary unit.
2. Compute the interest in the terms of that monetary unit.
3. Reduce the interest to its proper exchange units.

Example.—Find the interest on £200 5s. 6d. for 63 days at 6%.

Solution.

Pounds	£200.
5 shillings = ¼ £ (expressed decimally)	.25
6 pence = 1/40 £ " "	.025
Total pounds	£200.275
Interest for 60 days (Point off 2 places)	£ 2.0027
Interest for 3 days (1/20 of the interest for 60 days)	.1001375
	£ 2.1028375

Reducing the decimal of a pound to shillings, .1028375 × 20 = 2.05675 shillings.
Reducing the decimal to pence, .05675 × 12 = .67 pence.

Answer: Interest £2 2s. 1d.

In foreign exchange three days of grace are generally allowed, interest being calculated on the 365-day basis.

Problems.

Find the interest on:

1. £500 10s. 6d. at 6% for 93 days.
2. £1250 15s. 10d. at 5% for 93 days.
3. 3600 pesa at 6% for 63 days.
4. 2450 francs at 5% for 33 days.
5. 1050 yen at 6% for 93 days.
6. 5650 kroner at 4% for 63 days.

To find the value of time bills of exchange.—The four elements entering into the value of a time bill of exchange are:

1. Current rate of demand exchange.
2. Interest on money (exact time).
3. Stamp charges.
4. Commission of home or foreign banks.

Example.—If the demand rate of exchange is $4.46⅜, and the discount rate is 4%, stamps 1/20%, and commission ¼%, what is the 60-day rate?

Solution.

Demand rate		$4.46375
Deduct:		
Interest at 4% for 63/365 of year	.03081	
Stamps 1/20%	.00223	
Commission ¼%	.01116	.04420
60-day rate		$4.41955

Problems.

1. With the following facts, find the cost of a 60-day sight draft on London for £150 10s.

Demand rate on London	$3.90
London discount rate for 60-day sight draft	7%
Stamp charges	1/20%
Commission charged by English bank	1/4%
Commission charged by American bank	1/2%

2. The facts are:

Demand rate on Paris	$.0737
Discount rate 90 days sight	6%
Commission American banker	1/2%
Commission Paris banker	1/4%

What is the 90-day sight rate of exchange?

3. Mr. Davis desires to know the face value of an exchange on London which he can buy for $2500, under the following conditions:

Demand rate	$4.46375
90-day sight—discount rate	6%
Stamp charges	1/20%
English commission	1/4%
American commission	1/2%

4. The First National Bank of New York has received for collection a 60-day sight bill, drawn by a Brazilian exporter on a local coffee merchant, for $15,000, with the instructions that the proceeds shall be invested immediately in a demand draft on London, and forwarded to a London agent for the credit of the Brazilian exporter. The New York discount rate is 6% for 60-day sight bills (360-day basis, and no days of grace), and the New York Bank charges for collection 1/4% of the face of the bill. If the rate for demand drafts on London is $3.80½, what amount will be placed to the credit of the Brazilian exporter?

Would the purchase of a 90-day sight bill at $3.76 have been more advantageous if the London discount rate for a 90-day ($^{93}/_{365}$ of a year) sight bill was 5%, stamp charges 1/20%?

Foreign exchange accounts.—The American dealer with foreign countries keeps accounts with merchants abroad, and as these accounts are settled through banks of countries

where the exchange is fluctuating, it is necessary to establish a method of keeping records of converted transactions. The common practice is to enter the domestic value at the stated number of dollars, and the foreign value at an amount determined by converting to foreign values at the current rate. The balance of the account at closing time is converted either at cost or current buying rate. The shortage or excess is the loss or profit on exchange.

Example.—State the balance of the following account in both foreign and domestic currency, and show the profit or loss on exchange. Current price of exchange is quoted at $4.51 on the last day of the month.

DAVIS & COMPANY—BANKERS, LONDON

Debits

Jan. 1 Remittance sight bill £1000 at $4.46⅜.
Jan. 10 Remittance sight bill £500 10s. at $4.49½.

Credits

Jan. 8 Drafts drawn £100 at $4.47.
Jan. 24 Cable £1100 10s. 6d at $4.52½.

Solution.

Debits		*Foreign*	*Rate*	*Domestic*
Jan. 1	Remittance	£1000	$4.46⅜	$4463.75
Jan. 10	Remittance	£ 500 10s.	$4.495	2249.75
Profit on exchange.....				66.27
		£1500 10s.		$6779.77

Credits		*Foreign*	*Rate*	*Domestic*
Jan. 8	Drafts	£ 100	$4.47	$ 447.00
Jan. 24	Cable	£1100 10s. 6d.	4.525	4979.88
	Balance	£ 299 19s. 6d.	4.51	1352.89
		£1500 10s.		$6779.77

Explanation.—The balance of the account is found by first deducting the smaller side of the account in foreign money from the larger side. In the above case, the smaller side amounts to £1200 10s. 6d., the larger side amounts to £1500 10s.; the balance is £299 19s. 6d.; second, by converting this balance into domestic currency at the current rate of exchange,—£299 19s. 6d. at $4.51 equals $1352.89.

The profit or loss on exchange is the difference between the total debit and total credit of the domestic currency columns.

There might be a question, in determining the correct valuation of the balance on hand, as to whether the £299 19s. 6d. should have been valued at the last buying price of \$4.49½ or at \$4.51, the current market price. This is purely a question of accounting and finance. The usual practice is to take the current rate on the last day of the month, when given; otherwise, the cost of the exchange on hand, if that can be found by inspection, will be acceptable.

Problems.

1. In the following, find the balance of the account in both foreign and domestic currency and profit or loss on exchange:

LONDON BANKERS

Debit:						
October 1	Balance	£ 500	10s. 4d.	at	\$4.37	
October 10	Remittance	11000		at	4.41⅝	
October 20	Collection	200	0s. 6d.	at	4.41½	
October 25	Discounts	196	5s. 6d.	at	4.44¾	
Credit:						
October 3	Cable	£ 200		at	\$4.39	
October 7	Draft Sight	400	10s.	at	4.40 5/16	
October 9	Draft Sight	300		at	4.40 7/16	
October 15	Cable	2000	15s. 6d.	at	4.42 5/16	
October 20	Cable	400		at	4.44	

Demand rate on October 31, \$4.44⅜.

2. Find the balance of account in both foreign and domestic currency, and profit and loss on exchange:

PARIS BANKERS

Debit:				
October 1	Balance	5000	francs	\$.0752
October 15	Remittance	3000	"	.0750
October 30	Remittance	10000	"	.0739
Credit:				
October 4	Draft Sight	3000	francs	\$.0750
October 7	" "	2000	"	.0749
October 18	" "	4000	"	.0739
October 25	Cable	4000	"	.07395

Demand rate on October 31, franc = \$.07395.

3. A dealer in foreign exchange finds from his books that he has had the following transactions in London exchange during a particular month, namely:

Exchange bought in local market:

Jan. 1 30-day bill, payable in London £300 at $4.75.
Jan. 15 Bill due at sight in London £2500 at $4.76.

Exchange sold in local market:

Jan. 5 Bill due in London at sight £1000 at $4.77.
Jan. 20 Cable transfer £2000 at $4.78.

Foreign correspondent's draft honored and paid:

Jan. 20 Bill at 30 days after sight accepted December 21, £500 at $4.78.

State how the balance of the account stands at the close of the month, and how much profit or loss has been derived from the transactions. (At January 31 the rate for cable transfers is $4.80.)

Is the profit or loss so stated final?—*American Institute Examination.*

Averaging accounts in foreign exchange.—In general, the transactions in foreign exchange accounts are much larger and involve the holding of much more money than the average domestic account; because of this, interest is a very important factor. If the interest is not taken care of at the time of the transaction, it must necessarily be considered later.

The principle of averaging accounts, as explained previously in this text, will be applied here.

Example.—Find the average due date of the following account, also the amount due June 1, 1923, including interest at 6%:

Debits		*Credits*	
Jan. 2	£600	Feb. 15	£500
Jan. 31	£300	Apr. 1	£200
Mar. 16	£100		

Solution.

	Debits	*Days*	*Day £'s*	*Total*
Jan. 2	£600	0	0	
Jan. 31	£300	29	8700	
Mar. 16	£100	73	7300	
	£1000			16000
	Credits			
Feb. 15	£500	44	22000	
Apr. 1	£200	89	17800	
	£700			39800
Balance	£300	Bal. Day £		23800

Using the first day as the focal date, $23800 \div 300 = 79$ days. As the balance of the £'s and Day-£'s are on opposite sides, the due date will be counted backward from the focal date, that is, £300 should have been paid 79 days before January 2, or on October 15th of the previous year, in order that neither party should lose interest.

From October 15 of one year to June 1 of the next year is 229 days. £300 at 6% interest for 229 days is:

$$300 \times .06 \times \frac{229}{365} = £11.29315.$$

$$.29315 \times 20 = 5.8630 \text{ shillings.}$$

$$.863 \times 12 = 10.356 \text{ pence.}$$

Answer:

Due Aug. 15th last			
Principal	£300		
Interest to Jun. 1.....	11	5s.	10d.
Total due Jun. 1.....	£311	5s.	10d.

Problems.

1. Find the average due date of the following account, and the amount due on July 1, using 5% interest.

Debits				*Credits*			
Jun. 1.....	£300			Jun. 4.....	£500	15s.	9d.
Jun. 4.....	£200	10s.	6d.	Jun. 12.....	£300		
Jun. 6.....	£400			Jun. 24.....	£400	15s.	
Jun. 12.....	£700	15s.	3d.	Jun. 30.....	£700		
Jun. 25.....	£600						
Jun. 28.....	£500						

2. *A*, in London, in current account with *B* of New York, engages an accountant to prepare a statement, to be mailed to *B*, from the following data:

Debits			*Credits*		
1914			1914		
May 12.....	£750		Jun. 10.....	£500	
May 30.....	117		Jun. 30.....	300	
Jun. 12.....	340		Total Credits		800
Jul. 1.....	150		Balance		£ 557
Total Debits		£1357			

Find the average due date of the account and the interest at 5% to July 1, using 365 days to the year.

Conversion of foreign branch accounts.—The conversion of foreign branch accounts requires the application of different rates of exchange.

Current asset and current liability values should be converted at the rate prevailing as of the date of the balance sheet.

Fixed asset values should be converted at the rate prevailing at the time of purchase, or at the average rate for purchases during a fiscal period. If there have been no changes in fixed assets during the fiscal period, they should be valued at the same rate as before. Differences in values of fixed assets from one period to the other, due to fluctuations in exchange, should not be allowed to affect the results of the fiscal period.

Remittances should be converted at the actual rate on the day of the transmittal of the money.

Revenue items are to be converted at the average rate for the period.

The controlling or adjustment account, and the old inventory account should be converted at the rate established on the head office books as of the end of the last fiscal period.

An illustrative problem using the above principles of conversion is given.

Example.—A corporation, with its head office in Boston, has a branch office in London. The trial balances of the head office and the branch office on December 31, 1923, are:

BOSTON OFFICE

TRIAL BALANCE

	Dollars	*Dollars*
Cash	100,000	
Branch Account	876,000	
Remittance		130,500
Expenses	40,000	
Income		60,000
Capital Stock		800,000
Surplus		25,500
	1,016,000	1,016,000

LONDON BRANCH OFFICE

TRIAL BALANCE

	£	£
Cash	15,000	
Remittance	30,000	
Customers	130,000	
Inventory, December 31	50,000	
Expenses	10,000	
Income		25,000
Creditors		10,000
Boston Office Control		200,000
	235,000	235,000

An analysis of the remittance account showed:

Jun. 1 from London Branch	£10,000 @ 4.42
Sep. 1 " " "	10,000 @ 4.28
Dec. 1 " " "	10,000 @ 4.35

Current Rate	4.40
Average	4.30

From the above show:

1. Statement of conversion.
2. Consolidated trial balance working paper.
3. Consolidated profit and loss statement.
4. Consolidated balance sheet.

Solution.

(1)

BRANCH OFFICE

STATEMENT OF CONVERSION

	Pounds	*Pounds*	*Rate*	*Dollars*	*Dollars*
Cash	15,000		@ 4.40	66,000	
Remittances	30,000		@ 4.35	130,500	
Customers	130,000		@ 4.40	572,000	
Inventory	50,000		@ 4.40	220,000	
Expenses	10,000		@ 4.30	43,000	
Income		25,000	@ 4.30		107,500
Creditors		10,000	@ 4.40		44,000
Boston Control		200,000	@ 4.38		876,000
Profit on Exchange					4,000
	235,000	235,000		1,031,500	1,031,500

(2)

CONSOLIDATED TRIAL BALANCE WORKING PAPER AS OF DECEMBER 31, 1923

	Boston Office		*London Branch*		*Eliminations*		*Consolidated Trial Balance*	
Cash	$ 100,000		$ 66,000				$ 166,000	
Branch Account	876,000					$876,000		
Remittances from Branch		$ 130,500			$130,500			
Expenses at Home Office	40,000						40,000	
Income at Home Office		60,000						$ 60,000
Capital Stock		800,000						800,000
Surplus		25,500						25,500
Remittances to Home Office			130,500			130,500		
Customers			572,000				572,000	
Inventory			220,000				220,000	
Expense at Branch			43,000				43,000	
Income at Branch				107,500				107,500
Creditors				44,000				44,000
Boston Control				876,000	876,000			
Profit on Exchange				4,000				4,000
	$1,016,000	$1,016,000	$1,031,500	$1,031,500	$986,500	$986,500	$1,041,000	$1,041,000

(3)

CONSOLIDATED PROFIT AND LOSS STATEMENT

Income:		
Boston Office	$ 60,000	
London Office	107,500	
Total Income		$167,500
Expenses:		
Boston Office	40,000	
London Office	43,000	
Total Expenses		83,000
Operating Income		$84,500
Profit on Exchange *		4,000
Net Profit to Surplus		$88,500

* Some may question the advisability of including the profit on exchange in the net profits. Some accountants advise the setting up of an account "Reserve for fluctuation of exchange." This is purely an accounting problem, and will not be discussed here.

(4)

CONSOLIDATED BALANCE SHEET

Assets			
Cash:			
Boston		$100,000	
London		66,000	$166,000
Customers			572,000
Inventory			220,000
			958,000
Liabilities and Capital			
Creditors			$ 44,000
Capital			
Capital Stock		$800,000	
Surplus:			
Balance Jan. 1	$25,500		
Net Profit	88,500	114,000	914,000
			$958,000

In the above problem no fixed assets were given. Because this item sometimes appears in the Branch Trial Balance, other difficulties are encountered.

It is necessary in making a statement of conversion to divide the account on the Head Office books, "Branch Con-

trol," into "Branch Control Fixed Assets," and "Branch Control Current Assets." The rate at the time of purchase should be used for the conversion of the "Branch Control Fixed Assets" account. By converting it at this rate, the value of the fixed assets is found in terms of the monetary unit of the country in which the Head Office is located. By deducting the value of the fixed assets, or "Branch Control Fixed Assets" from the "Branch Control" account, the value of the "Branch Control Current Assets" is given.

To illustrate this the following example is given:

Example.—A United States company, doing business in London, through its London Branch received a trial balance of the branch on December 31, as follows:

LONDON BRANCH

	£	£
Cash	5,000	
Remittances	25,000	
Customers	50,000	
Inventory (new)	25,000	
Expenses	25,000	
Plant	100,000	
Creditors		50,000
Income from Sales		50,000
New York Control		130,000
	230,000	230,000

NEW YORK OFFICE

TRIAL BALANCE

Cash	$100,000	
London Branch	617,500	
Remittances		$118,875
Expenses	50,125	
Capital Stock		500,000
Surplus		148,750
	$767,625	$767,625

Remittances were received as follows: £5000 at $4.75, £5000 at $4.75¼, £5000 at $4.75½, £5000 at $4.75¾, and £5000 at $4.76. The current rate of exchange was $4.76½, the average rate of exchange was $4.75¼, and the fixed rate of exchange for the plant was $4.74½.

From the information given, show (*a*) a statement of conversion; (*b*) a balance sheet of the London Branch, and (*c*) a consolidated balance sheet of the New York Office, as well as (*d*) the accounts on the New York Office books taking up the profit.

Solution.

STATEMENT OF CONVERSION

	£	£	Rate	$	$
Cash	5,000		$4.76½	23,825.00	
Remittances	25,000		4.75½	118,875.00	
Customers	50,000		4.76½	238,250.00	
Inventory	25,000		4.76½	119,125.00	
Expense	25,000		4.75¼	118,812.00	
Plant	100,000		4.74½	474,500.00	
Creditors		50,000	4.76½		238,250.00
Income from Merchandise		50,000	4.75¼		237,625.00
New York Control (Plant Acc't)		100,000	4.74½		474,500.00
New York Control (Current Acc't)		30,000	4.76⅔		143,000.00
	230,000	230,000			
Profit on Exchange					12.50
				1,093,387.50	1,093,387.50

CONVERSION OF BRANCH CONTROL ACCOUNT

PLANT AND FIXED ASSETS

London Branch control		$617,500
Deduct:		
Value of Plant	£100,000	
Multiplied by rate of exchange for fixed assets	4.74½	
Value of fixed assets in dollars	$474,500	
		474,500
		$143,000
New York control	£130,000	
Less Plant account	£100,000	
Current assets at Branch	£ 30,000	
143,000 ÷ 30,000 = rate of exchange of current assets	4.76⅔	
	$143,000	$143,000

BALANCE SHEET OF THE LONDON BRANCH

Assets	£	*Rate*	*Dollars*
Cash	5,000	$4.76½	23,825
Customers	50,000	4.76½	238,250
Inventory	25,000	4.76½	119,125
Plant	100,000	4.74½	474,500
	180,000		855,700
Liabilities			
Creditors	50,000	$4.76½	238,250
Capital	130,000		617,450
Plant	100,000	$4.74½ = $474,500	
Current (Schedule A)	30,000	4.76½ = 142,950	617,450

SCHEDULE A

CURRENT ACCOUNT

Credit	£	*Rate*	*Dollars*	
Balance	30,000	$4.76⅔	143,000.00	
Income	50,000	4.75¼	237,625.00	
	80,000			
Profit on Exchange			12.50	
				$380,637.50
Debits				
Expenses	25,000	$4.75¼	118,812.50	
Remittances	25,000	4.75½	118,875.00	
	50,000			237,687.50
Balance	30,000			$142,950.00

$142,950 ÷ £30,000 = $4.76½ Rate.

NEW YORK OFFICE

LONDON CONTROL ACCOUNT

Bal.	Plant Account	$474,500.00	Remittance		$118,875.00
	Current "	143,000.00	Expense		118,812.50
Income from Merchandise		237,625.00	Bal.	Plant Account	474,500.00
Profit on Exchange		12.50		Current "	142,950.00
		$855,137.50			$855,137.50

PROFIT AND LOSS ACCOUNT

Expense of Branch	$118,812.50	Income of Branch	$237,625.00
Expense of Home Office	50,125.00	Profit on Exchange Branch	12.50
Profit to Surplus	68,700.00		
	$237,637.50		$237,637.50

NEW YORK BALANCE SHEET

Assets				
Current:				
Cash				
New York	$100,000			
London	23,825	$123,825		
Customers—London		238,250		
Inventory—London		119,125	$481,200	
Fixed:				
Plant—London			474,500	$955,700
Liabilities				
Current:				
Creditors—London				238,250
				$717,450
Capital:				
Capital Stock			$500,000	
Surplus		148,750		
Net Profits		68,700	217,450	
				$717,450

A balance sheet of the London Branch is next set up. This balance sheet needs no comment, except to call attention to the schedule of the current account, and the method of finding the rate.

Various methods of taking up the profits of the branch on the Head Office books are used. Probably no comment need be made, except as to the credit to Profit and Loss account of the $12.50 profit on exchange. This might have been credited to an account called "Reserve for Fluctuation of Exchange." It is not our purpose to discuss the theory of accounting, but so far as possible confine the discussion to the mathematics of accounting.

Problems.

1. The trial balance of the London Office books of the *A* Rubber Company is as follows, on December 31, 1909:

	£	s.	d.	£	s.	d.
Estate Purchase	3,000	0	0			
Estate Development	8,000	0	0			
Estate Produce Stock, 1st Jan. 1909	600	0	0			
Cash at Bank, London	800	0	0			
Estate Manager, 1st Jan. 1909	500	0	8			
Remittance to Estate Manager	1,000	0	0			
London Office Expenses	400	0	0			
Share Capital				12,000	0	0
Creditors				1,900	0	0
Profit and Loss Balance				400	0	8
	£14,300	0	8	£14,300	0	8

After taking out the above balances, the accounts to December 31, 1909, are received from the estate manager as follows (the dollar to be taken at 2s. 4d.):

Balance, 1st Jan. 1909	$ 4,286	
Remittance from London	8,600	
Rebates	131	
Sale of Produce	2,000	
Profit on Rice	249	
Expenditure on Development		$ 9,000
Expenditure on Purchase of New Land		2,800
Expenditure on Upkeep of Estate		1,646
Balance Carried Forward		1,820
	$15,266	$15,266

The produce unsold at December 31, 1909, was valued by the manager at $5500.

You are required to construct the revenue account and balance sheet for presentation to the shareholders.—*Chartered Accountants' Examination.*

2. Wright, Dunbar & Company, of New York, U. S. A., and Van Allen & Company, of Amsterdam, Holland, bankers and dealers in foreign exchange, enter into a joint venture on January 2, 1914, for the purpose of dealing in foreign exchange.

It is agreed that profits or losses are to be shared equally, that interest on the account current is to be figured at 6% per annum, exact number of days per month, and that guilders at sight are to be calculated at 40¼.

The blotter of the New York bankers records the following completed transactions for the joint venture of the first month's operations:

January 10, 1914.
Received on account from Borton Bros., New York, and sent to Van Allen & Company, account joint venture, £2000 on London, due February 10, 1914, at 4.87 sight.

January 16, 1914.
Received from Van Allen & Company, value January 2, 1914, their draft on Wiener Bros., due February 25, 1914, kronen 48,000 on Vienna at 49½ guilders.

January 16, 1914.
Cable from Van Allen & Company that they have sold for account joint venture, value this day, £2000 on London, due February 10, 1914, at 12 guilders.

January 16, 1914.
Discounted this day at National City Bank, New York, for account joint venture, value January 16, 1914, kronen 48,000 on Vienna, due February 25, 1914, at 20¼.

January 27, 1914.
Received from Van Allen & Company value January 13, 1914, their draft on Charles & Company, in Berlin, due February 13, 1914, Rm. 30,000 at 59 guilders.

January 28, 1914.
Sold for cash account joint venture, value this day, Rm. 30,000 on Berlin, due February 13, 1914, at $.238.

Prepare as at January 31, 1914:

(*a*) A statement showing separately the results of operations as conducted in New York and Amsterdam, respectively;

(*b*) The ledger accounts of the joint venture and of Van Allen & Company as kept by the New York firm of bankers.—*C. P. A.—New York.*

3. Transpose into dollars and cents, the following items of a London Branch:

Fixed Assets	£5000
Inventory (new)	£1000
Cash	£1000
Profit and Loss of Period	£1000
Head Office Account	£5000
Remittance to Head Office	£1000

RATES OF EXCHANGE

Current rate	$4.78½
Average remittance rate	4.62$^{3}/_{16}$
Rate at opening of business	4.86$^{3}/_{16}$
Average rate during year	4.60$^{7}/_{16}$
Balance on Head Office books	$27,200

Part II

Chapter 27

COMPOUND INTEREST

Compound interest.—Compound interest is usually defined as "interest on interest." This definition is not technically true, because compound interest contains that which is termed "Simple interest" as well as "interest on interest." By reference to the table on page 219, the verification of the above statement can readily be seen.

Legality of compound interest.—The question is often raised—"Is compound interest legal?" Some courts have held that it is not, while others have held that it is. However, compound interest is justifiable from a mathematical standpoint. Let us suppose an individual loans money to one person, and that when the first interest is due, he collects the interest and immediately loans the money collected to a second person at the same rate of interest. At the end of the second year the interest is again collected and immediately put out at interest, and so on, each year. The individual has in effect put his money out at compound interest. The effect of the mathematical calculation is the same as though he receives compound interest.

From the investor's point of view, no more accurate and scientific method may be employed to find the true value of an investment than the compound interest method.

The mathematical theory of investment is based on the assumption that all accumulations of interest become a part of the investment at the end of each interest period.

Relation of compound interest to actuarial science.—Actuarial science is the mathematical science of compound interest, and insurance probabilities. It deals with the investment of funds as well as the mortality tables for insurance companies. Compound interest is one of the principal bases on which actuarial science is founded.

The actuary.—An actuary is one who is skilled in the mathematics of actuarial science. He uses tables for the greater part of his work; yet he must have a knowledge of the fundamentals of his science. He must be able to apply all the principles of his science to finances of the insurance companies so far as they affect investments, bonds, and long-time accumulations of money.

Is the accountant interested in the study of actuarial science?—The accountant is interested in actuarial science only so far as his work is affected. He probably will never be called upon to formulate tables for insurance companies, and may not often be called upon to do the work of an actuary in large companies. But if he gives the best service to his client he must be able to give definite information on the amount an investment will produce in a given time, or how much must be set aside in a fund to meet a bond issue due in the future, and must be able to answer many other similar questions. Actuarial science, as used in this work, will pertain only to the needs of the accountant.

Symbols.—Much of the misunderstanding and confusion in the study of the mathematics of investment accounting is caused by the lack of uniformity of symbols. Indeed, it has been found that three or more different symbols are in use to describe some particular phase of an expression.

Considering the numerous combinations which these variations will make in a formula of from three to eight symbols, there is slight wonder that the student of accounting is bewildered. In the choice of symbols, the endeavor has been made to select those which are in the most common use, and follow their use consistently throughout the work.

The following symbols are assembled for constant reference:

1	A unit of value, as \$1, or basis of any unit of value.
i	The rate of interest for a single period.
j	The nominal annual rate, if interest is compounded oftener than once each year.
n	The number of periods.
r	The periodic ratio of increase, or $(1+i)$.
m	Frequency of periods during year.
a	The amount of \$1.

c The cash rate of interest for a single period.
d The rate of discount.
p The present worth of $1.
D The compound discount on $1.
I The compound interest on $1.
A The present worth of a $1 annuity (*"V"* is also used).
R Rent, or periodic payment of an annuity.
S Amount of an annuity of $1.
P The principal.

Formulas.—Formulas, like symbols, are the cause of much bewilderment, and to a person who has not had higher mathematics, they are a never-ending source of trouble.

The authors have endeavored to be consistent in the use of symbols in the formulas, and to add to each previous formula the necessary step to show the result required. This care obviously necessitates the use of long and seemingly complicated formulas, but by starting at the beginning and mastering each principle, the reader will find that they become *quite simple.*

Principal.—The principal is a sum of money or element of value for which interest is paid, or on which interest computations are based. 1 or $1 will be used in most calculations in this work.

Period of time.—The time which the investment has to run is commonly stated in years and months, but in compound interest computations it is better stated as the number of periods of equal duration over which the investment runs. If the problem calls for "four years and six months, interest compounded semiannually," the time should be stated as nine periods of six months each.

Illustration.

Time	*Frequency of Compounding*	*Number of Periods or Value of "n"*
1 yr.	Annually	1
1 "	Semiannually	2
1 "	Quarterly	4
1 "	Monthly	12
3 yrs. 6 mos.	Semiannually	7
3 " 6 "	Quarterly	14
5 "	Annually	5
5 "	Monthly	60

Rate.—The rate is the measure of increase of value of, or interest on, the investment or principal. It is often indicated in different ways, as .06, 6%, or $\frac{6}{100}$. The rate is generally stated as so much a year, but when the period is of any length other than a year, it is necessary to state the rate as *so much per period.*

If the interest period is semiannual, it is necessary to divide the stated annual rate by two, and multiply the time in years by two; or if the interest period is stated quarterly, divide the annual rate by four and multiply the time in years by four.

Illustration.

Time	*Rate*	*Frequency of Compounding*	*Number of Periods or Value of "n"*	*Rate or Value of "i"*
1 yr.	6%	Annually	1	6%
1 "	6%	Semiannually	2	3%
1 "	6%	Quarterly	4	1.5%
1 "	6%	Monthly	12	0.5%
2 yrs.	6%	Semiannually	4	3%
2 "	6%	Monthly	24	0.5%
3 yrs. 6 mos.	8%	Semiannually	7	4%
3 " 6 "	4%	"	7	2%
3 " 6 "	4%	Quarterly	14	1%

Ratio of increase.—From every investment, the business man expects to receive in return the amount of his investment plus interest. He buys bonds, stocks, or other investments with the expectation of receiving an increased amount in return. He cannot say that one dollar invested in bonds will bring in a greater income than any other dollar of the same investment; hence, if the total investment be multiplied by 1 plus the interest rate expressed decimally, as 1.06, the result will be the amount of his investment at the end of the period. The 1 plus the interest rate is called the "ratio of increase."

Again, if the principal, $1, is placed at interest at 6% for one year, it will be worth $1.06 at the end of the year. The amount to be added to the $1 is $.06. The ratio of increase is expressed as $(1 + .06)$ or 1.06; by using the symbols given before, it would be written in the algebraic formula $(1 + i)$ or r.

Compound amount tables.—A compound amount table is a compilation of products of the value of 1 for various

numbers of periods at various rates per cent. It is based on the ratio of increase, using the per cent as the rate of increase, and n as the number of periods.

A compound amount table is given in Appendix B.

When a compound amount table is available, many calculations may be eliminated. However, if one is not available or the factors to be used are not given in the table, the desired amount may be obtained by one of the methods which follow.

Calculation of compound amount.—The following methods of calculation are given to show the different ways of arriving at the value of $1 for a given time.

Method 1.—Method 1 shows each step taken to find the value at the end of each period.

Example.—Find the value of $1 at 6% compound interest for 8 years.

Solution.

$$
\begin{aligned}
1.00 \times 1.06 &= 1.06 \\
1.06 \times 1.06 &= 1.1236 \\
1.1236 \times 1.06 &= 1.191016 \\
1.191016 \times 1.06 &= 1.262477 \\
1.262477 \times 1.06 &= 1.338225 \\
1.338225 \times 1.06 &= 1.418519 \\
1.418519 \times 1.06 &= 1.503630 \\
1.503630 \times 1.06 &= 1.593848
\end{aligned}
$$

It can be seen that the compound amount, 1.593848 is the sum of the investment and the compound interest.

The above method of arriving at the compound amount becomes very cumbersome when there are many periods.

Method 2.—In this method, multiplication is performed by using powers of numbers.

Example.—Find the value of $(1+i)^{12}$ or $(1.06)^{12}$.

Solution.

ALGEBRAIC INCREASE

$$
\begin{aligned}
(1+i) \times (1+i) &= (1+i)^2 \\
(1+i)^2 \times (1+i)^2 &= (1+i)^4 \\
(1+i)^4 \times (1+i)^4 &= (1+i)^8 \\
(1+i)^8 \times (1+i)^4 &= (1+i)^{12}
\end{aligned}
$$

ARITHMETICAL INCREASE

Exponents Added		*Powers of Ratio of Increase Multiplied*
(1.06)	=	1.06
(1.06)	=	1.06
$(1.06)^2$	=	1.1236
$(1.06)^2$	=	1.1236
$(1.06)^4$	=	1.262477
$(1.06)^4$	=	1.262477
$(1.06)^8$	=	1.593848
$(1.06)^4$	=	1.262477
$(1.06)^{12}$	=	2.012196

The above principle may be applied for any power of a number. If the compound interest table does not extend to a sufficient number of periods, the required value may be found by the method illustrated in the following example:

Example.—Find the compound amount of \$1 at 6% for 80 years.

Solution.

Assuming that we referred to the Compound Interest Table at hand, and found that the highest value shown at 6% was for 20 years, and was 3.2071355, the calculation for 80 years could be made as follows:

Exponents Added		*Powers of Ratio of Increase Multiplied*
$(1.06)^{20}$	=	3.2071355
$(1.06)^{20}$	=	3.2071355
$(1.06)^{40}$	=	10.285718
$(1.06)^{40}$	=	10.285718
$(1.06)^{80}$	=	105.795994

Method 3.—Calculation by the third method is made by the use of logarithms.

Example.—Find the value of $(1.06)^{80}$.

Solution.

Log 1.06	=	0.0253059
Multiply by power	=	80
Log of power	=	2.0244720
Antilog 2.024472	=	105.80

The use of a compound interest table, if one is at hand, saves time and makes most of the above calculations unnecessary.

Problems.

Work the following problems by the methods indicated, and check your results by the compound interest table. Show the complete work as in above solutions.

1.	Find	the	compound	amount	of	$(1.01)^5$	by	Method	1.
2.	"	"	"	"	"	$(1.03)^6$	"	"	1.
3.	"	"	"	"	"	$(1.06)^8$	"	"	1.
4.	"	"	"	"	"	$(1.02)^{12}$	"	"	2.
5.	"	"	"	"	"	$(1.03)^{16}$	"	"	2.
6.	"	"	"	"	"	$(1.06)^{20}$	"	"	2.
7.	"	"	"	"	"	$(1.05)^{40}$	"	"	2.
8.	"	"	"	"	"	$(1.04)^{40}$	"	"	3.
9.	"	"	"	"	"	$(1.03)^{34}$	"	"	3.
10.	"	"	"	"	"	$(1.02)^{46}$	"	"	3.

Compound amount.—To compute the compound amount of any principal apply the following procedure:

(*a*) Compute the compound amount of 1 for the number of periods at the given rate.

(*b*) Multiply the compound amount of 1 by the number of dollars in the investment.

Example.—What will be the compound amount of $100 placed at interest at 6% for four years, interest compounded annually?

Algebraic Formula.	*Arithmetical Substitution.*
$P(1+i)^n$ = Compound amount.	$\$100\ (1.06)^4$ = Compound amount.

Extended Solution.

1.00	$\times$ 1.06	=	1.06,	or amount of 1 for 1 year = $(1.06)^1$.
1.06	$\times$ 1.06	=	1.1236,	" " " " 2 " = $(1.06)^2$.
1.1236	$\times$ 1.06	=	1.1910,	" " " " 3 " = $(1.06)^3$.
1.1910	$\times$ 1.06	=	1.2625,	" " " " 4 " = $(1.06)^4$.
$100.00	$\times$ 1.2625	=	$126.25,	compound amount of $100 for 4 years.

Hereafter, instead of finding the compound amount of 1 at the end of each year as above, the solutions will be shortened by using the value of the expression, $(1+i)^n$, as in the following:

Solution (Contracted).

$(1.06)^4 = 1.2625$, compound amount of 1 for four years.
$\$100 \times 1.2625 = \$126.25$, compound amount of \$100 for four years.

Reference to the compound interest table is necessary in the remainder of this work.

Problems.

Construct formulas and find (contracted) solutions for the following:

1. For \$ 500 at 6% compound interest for 6 years.
2. " \$ 200 " 3% " " " 10 "
3. " \$1000 " 7% " " " 4 "
4. " \$ 150 " 4½% " " " 5 "
5. " \$ 250 " 3½% " " " 10 "

Compound interest.—Since the investment placed at interest for a definite time at a fixed rate will produce a given amount, the difference between the amount at the end of the time and the investment at the beginning of the time will be the increase, or compound interest. To compute the compound interest, it is therefore necessary to find the amount at the end of the time and deduct from it the principal.

Procedure.—(*a*) Determine the compound amount of 1 for the number of periods at the given rate.

(*b*) Find the compound interest by deducting 1 from the compound amount of 1.

(*c*) Multiply the compound interest on 1 by the number of dollars in the investment.

Example.—Find the compound interest on \$100 for 4 years at 6%.

Algebraic Formula.	*Arithmetical Substitution.*
$P[(1+i)^n - 1]$ = Compound interest.	$100[(1.06)^4 - 1]$ = Compound interest.

Solution.

$(1.06)^4 = 1.2625$, compound amount of 1 for 4 years.
$1.2625 - 1 = .2625$, compound interest of 1 for 4 years.
$\$100. \times \$.2625 = \$26.25$, compound interest of \$100 for 4 years.

TABLE OF ANALYSIS OF COMPOUND INTEREST

1 *End of Period*	*2* *Principal*	*3* *Simple Interest*	*4* *Interest on Interest*	*5* *Compound Interest*	*6* *Compound Amount*
1.	1.00	.06		.06	1.06
2.	1.00	.12	.0036	.1236	1.1236
3.	1.00	.18	.011016	.191016	1.191016
4.	1.00	.24	.022477	.262477	1.262477

Problems.

1. Give formulas and computations of the compound interest on $300 if invested at 4% for 72 years.

2. Construct a table similar to the above for $500 at 4% interest for 5 years.

3. Give the formulas and the solutions, and construct a table of analysis similar to the above, showing both simple and compound interest on $400 at 5% for 5 years.

Results of frequent conversions of interest.—Compound interest is usually stated at a certain rate per annum, but if the interest is to be compounded more often than once each year, the total accumulation will be greater than the accumulation of $(1 + i)^n$ times the principal, using "i" as the annual rate.

A study of the following will show the difference between the amounts of an investment of $100 at 6% interest, compounded monthly, and of $100 at 6% simple interest.

$$100 \times (1 + i)^n \text{ or } (1.005)^{12} = 106.168$$
$$100 \times (1 + i) \text{ or } (1.06) = 106.$$
$$\text{Difference caused by compounding} = .17$$

While the difference in the interest in the above problem is small, the increase in interest would be large with a large investment running for a long term of years.

Nominal and effective rates.—Nominal, as the word implies, is defined as, "in name only." In the preceding illustration, six per cent is the nominal rate. Effective interest is interest which is actually received by the investor based upon the amount invested and *one year as the period of time.* In the preceding illustration, $6.168 is the amount of interest received in one year on an investment of $100 which, in effect, is 6.168% on the investment, or an effective rate of 6.168%.

Procedure in calculation of the effective rate.—

(*a*) Find the rate per period.

(*b*) Find the number of periods per year.

(*c*) Determine the compound amount of 1 for the number of periods (*b*), and at the rate per period (*a*).

(*d*) Deduct 1 from the compound amount of 1.

Example.—What is the annual effective rate of $1 at 6%, compounded quarterly?

Arithmetical Statement.

$$\left(1+\frac{.06}{4}\right)^4-1=\text{Effective rate.}$$

Solution.

$.06 \div 4$	= .015, rate per period.
$1. \times 4$	= 4, number of periods.
$1. \times 1.015$	= 1.015, compound amount of 1 for 1 period at 1.5%.
1.015×1.015	= 1.030225, compound amount of 1 for 2 periods at 1.5%.
1.030225×1.030225	= 1.061364+, compound amount of 1 for 4 periods at 1.5%.
$1.061364 - 1$	= .061364+, or 6.1364% annual effective rate.

Procedure in calculation of the nominal rate, the effective rate being given.—

(*a*) Determine the log of 1 plus the effective rate, or the effective ratio of increase.

(*b*) Divide the log found in (*a*) by the number of periods of compounding.

(*c*) Find the antilog of the quotient of (*b*) to determine the ratio of increase.

(*d*) Deduct 1 from the ratio of increase.

Example.—If an Insurance Company receives 6% effective interest on a certain investment, what is the nominal rate per annum if converted quarterly?

Arithmetical Statement.

$$\left(\sqrt[4]{(1.06)}-1\right)4=\text{Nominal rate.}$$

Solution.

Log (1.06)	= 0.0253059.
$0.0253059 \div 4$	= 0.0063265, log of ratio of increase.
Antilog 0.0063265	= 1.014672, ratio of increase.
$1.014672 - 1$	= .014672, rate of increase for 1 quarter.
$.014672 \times 4$	= .058788 or 5.8788%, nominal rate per annum.

Note.—"$\sqrt[4]{(1.06)}$" is sometimes written $(1.06)^{\frac{1}{4}}$.

Semiannual interest.—While the effective and nominal rate calculations are always based on a unit period of a year, most investments are for periods of more than one year's time. The added feature of a number of years may be taken care of by either of two methods:

(*a*) Find the effective rate for the year and use the actual number of years for which the investment has to run.

(*b*) Divide the nominal rate per cent by the number of times the interest is to be compounded each year, and multiply the time in years by the same figure used in dividing the interest, using these results as the rate and time.

The use of the first or effective interest method is advantageous in some annuity computations.

The second method is the one used most commonly because of its simplicity, and will be used hereafter unless it is necessary to use the first method for explanatory purposes.

Procedure.—(*a*) Find the rate per period.

(*b*) Find the total number of periods.

(*c*) Determine the compound amount of 1 for the number of periods at the rate found in (*a*).

Example.—What amount will be due in 4 years if $100 be put at interest at 6%, compounded each six months?

(*a*) The number of interest periods is $4 \times 2 = 8$.

(*b*) The rate of interest per period is $.06 \div 2 = .03$.

Algebraic Formula.	*Arithmetical Substitution.*
$P(1+i)^n$ = Amount.	$100\ (1.03)^8$ = Compound amount.

Solution.

$(1.03)^8$ = 1.2668, compound amount of 1 for 8 periods at 3%.
$100. × 1.2668 = $126.68, compound amount of $100 for 8 periods at 3%, or the amount at 6%, compounded semiannually for four years.

Example.—What is the compound amount of $100 at 6% for four years, interest payable quarterly?

(*a*) The number of interest periods is $4 \times 4 = 16$.

(*b*) The rate of interest per period is $.06 \div 4 = .015$.

Algebraic Formula.	*Arithmetical Substitution.*
$P(1+i)^n$ = Amount.	$100\ (1.015)^{16}$ = Amount.

Solution.

$(1.015)^{16} = 1.2689$, compound amount of 1 for 16 periods at 1.5%.
$\$100. \times 1.2689 = 126.89$, compound amount of \$100 for 16 periods at 1.5%, or for 4 years at the rate of 6%, compounded quarterly.

Problems.

Construct formulas and solutions for problems 1 and 2.

1. If John Brown deposits \$300 in the savings bank on January 1, 1920, how much will be due to him on January 1, 1930, if the bank pays 4%, interest compounded quarterly?

2. James Davis deposited \$400 in the bank on January 1, 1923, and left it there for five years. The bank paid 4%, interest compounded quarterly. How much interest was earned?

3. What is the effective rate, if the nominal rate is 5%, and the interest is compounded quarterly?

4. What effective rate is equivalent to a nominal rate of 7½%, convertible quarterly?

5. Construct a table, similar to the one shown under the subject of compound interest, for five years at 5%, interest compounded semiannually.

Compound present worth.—Sometimes it is desired to find *how much must be placed at interest now,* in order to amount to a certain given sum at a definite future time.

The present worth of a sum which is due at the end of a certain number of periods is a smaller sum which if put at compound interest at a given rate will amount to the known sum in the given time.

The ratio of increase employed in accumulating 1 to a compound amount is the same as the ratio of increase employed in accumulating a present worth to 1. To illustrate:

Compound Amount.		*Present Worth.*	
\$1.	basis of calculation	\$ .792094	present worth of \$1
1.06	multiplying	1.06	multiplying
1.06		.839619 +	
1.06	multiplying	1.06	multiplying
1.1236		.889996 +	
1.06	multiplying	1.06	multiplying
1.191016		.943396 +	
1.06	multiplying	1.06	multiplying
1.262477 +	compound amount	1.00000	basis of calculation

$1. \div 1.262477 = .792094$, present worth.
or
$1. \div .792094 = 1.262477$, compound amount.

Procedure.—(*a*) Compute the present worth by dividing 1 by the compound amount of 1.

(*b*) Multiply the present worth of 1 by the number of dollars to be produced.

Example.—What amount of money, invested at compound interest at 6% for four years, will produce $100?

Algebraic Formula.

$$\frac{1}{(1+i)^n} = \text{Present worth.}$$

Arithmetical Substitution.

$$100\ \frac{1}{(1.06)^4} = \text{Present worth.}$$

Solution.

$(1.06)^4 = 1.26247$, compound amount of 1 for 4 years at 6%.
$1 \div 1.26247 = .7921$, compound present worth of 1 for 4 years at 6%.
$\$100 \times .7921 = \$79.21$, compound present worth of $100 for 4 years at 6%.

Verification.

79.21 × 1.06 = 83.96, compound amount for 1 year.
83.96 × 1.06 = 89.00, " " " 2 years.
89.00 × 1.06 = 94.34, " " " 3 "
94.34 × 1.06 = 100.00, " " " 4 "

TABLE

1 *End of Period*	*2* *Principal*		*3* *Compound Amount (reversed)*		*4* *Present Worth of 1*	*5* *Compound Discount*
	1.00	÷	1.262477	=	.792094	.207906
1	1.00	÷	1.191016	=	.839619	.160381
2	1.00	÷	1.1236	=	.889996	.110004
3	1.00	÷	1.06	=	.943396	.056604
4	1.00	÷	1.00	=	1.00	

Problems.

Construct formulas, solutions, and verifications for problems 1 and 2.

1. What amount placed in the bank at 4% interest compounded annually will in 4 years amount to $600?

2. Mr. Evans desires to know how much money he will have to place at interest at 6%, compounded semiannually, so that in three years he may have $3000?

3. Construct a table, similar to the one on this page, for $1, at 5% for five years, interest compounded annually.

Compound discount.—The compound discount is the difference between 1 and the present worth of 1.

By reference to column 5 of the table on page 223 it will be observed that the compound discount increases as the present worth decreases.

Procedure.—(*a*) Calculate the compound discount by deducting from 1 the present worth of 1.

(*b*) Multiply the compound discount on 1 by the number of dollars.

Example.—What is the compound discount on $100 due in four years if money can be invested at 6%, compounded annually?

Algebraic Formula.

$$P\left[1-\frac{1}{(1+i)^n}\right]=\text{Compound discount.}$$

Arithmetical Substitution.

$$100\left[1-\frac{1}{(1.06)^4}\right]=\text{Compound discount.}$$

Solution.

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%.
$1 \div 1.262477 = .792094$, compound present worth of 1 due at the end of 4 years at 6%.
$1 - .792094 = .207906$, compound discount on 1 due at the end of 4 years at 6%.
$\$100 \times .207906 = \$20.79$, compound discount on $100 due at the end of 4 years at 6%.

Problems.

Construct formulas and find solutions in connection with the following:

1. Find the compound discount on $400, due in eight years, money being worth 6%.
2. Compute the compound discount on $900, due in five years, money being worth 4%.
3. Required, the compound discount on $600, due in ten years, money being worth 5%.

Rate.—The rate of interest may be computed if the principal, the amount, and the time are known. This process

involves the use of a principle and formula illustrated in the chapter on logarithms.

Procedure.—(*a*) Calculate the compound amount of 1 by dividing the given compound amount by the principal.

(*b*) Determine the log of the compound amount of 1.

(*c*) Divide the log of 1 by the number of periods.

(*d*) Determine the ratio of increase by finding the antilog of (*c*).

(*e*) Deduct 1 from the ratio of increase.

Example.—If $100 produces $126.25 in four years, what is the rate per cent?

Algebraic Formula.

$$\sqrt[n]{\frac{\text{Amount}}{\text{Principal}}} - 1 = i.$$

Arithmetical Substitution.

$$\sqrt[4]{\frac{126.25}{100.00}} - 1 = \text{Rate}.$$

Solution.

126.25 ÷ 100 = 1.2625, compound amount of 1 for 4 years at the unknown rate.
Log 1.2625 = 0.101231
0.101231 ÷ 4 = 0.025307
The antilog of 0.025307 = 1.06, ratio of increase.
1.06 — 1 = .06 or 6%, the rate.

Verification.

1.00 × 1.06 = 1.06
1.06 × 1.06 = 1.1236
1.1236 × 1.06 = 1.1910
1.1910 × 1.06 = 1.2625
$100 × 1.2625 = $126.25

Problems.

1. What is the annual rate in each of the following:

	Principal	*Amount*	*Time*	*Rate*
(a)	$ 100	$ 162.89	10 years	?
(b)	200	438.22	20 "	?
(c)	80	143.27	10 "	?
(d)	1000	2093.77	25 "	?
(e)	40	46.87	8 "	?

2. At what nominal rate per annum will $100 amount to $180.61 in ten years' time, if interest is converted semi-annually?

3. At what rate of interest, converted quarterly, will any principal double itself in nine years? (Accurate to 3 decimal places.)

Time.—By applying the principles of logarithms, the time may be computed if the principal, amount, and rate are given.

Procedure.—(*a*) Determine the compound amount of 1 by dividing the compound amount by the principal.

(*b*) Determine the log of the compound amount of 1.

(*c*) Determine the log of the ratio of increase.

(*d*) Divide (*b*) by (*c*) to determine the time in periods.

Example.—If \$100 placed at interest at 6% compounded annually produces \$126.25, what will be the time of investment?

Algebraic Formula.

$$\frac{\text{Log of } \frac{\text{Amount}}{\text{Principal}}}{\text{Log of } (1+i)} = n.$$

Arithmetical Substitution.

$$\frac{\text{Log of } \frac{126.25}{100.00}}{\text{Log of } (1.06)} = \text{Time.}$$

Solution.

126.25 ÷ 100 = 1.2625, compound amount of 1 at 6% for *n* periods.
Log 1.2625 = 0.101231
Log 1.06 = 0.025306
0.101231 ÷ 0.025306 = 4, time in periods.

Hence, the time is 4 years.

Problems.

1. Compute the time in the following:

	Principal	*Amount*	*Rate*	*Time*	*Convertible*
(a)	\$ 100	\$ 162.89	5%	?	Annually
(b)	1000	1338.22	6%	?	"
(c)	200	438.22	4%	?	"
(d)	40	65.62	2%	?	"
(e)	500	2400.51	4%	?	"
(f)	300	450.00	6%	?	Semiannually

2. How long will it take any principal to double itself at 6%, interest convertible semiannually? Give the answer in years and decimal of years.

Chapter 28

ORDINARY ANNUITIES

Definition.—An annuity is a series of payments of similar amounts made at equal intervals of time. Usually each of the periodic payments draws compound interest from the date of the payment until the final payment date of the annuity. The word "annuity" indicates annual payments, but the common meaning of the term is any series of payments at stated intervals, whether they be made annually, semiannually, quarterly, or at the end of any other period of time.

The principle is recognized that an individual is entitled to a return on his money each period, and when he receives this return he adds it to his second period's payment and immediately invests this amount. He may continue this process indefinitely.

Annuities are divided into two kinds as to time; first, *ordinary annuities,* the periodic payments of which are made at the *end* of each period; and second, *annuities due,* the payments of which are made at the *beginning* of each period.

Ordinary annuity.—The amount of an annuity is indicated by the symbol S.

Rent of an annuity.—The periodic payments are sometimes known as "rents."

Amount of an ordinary annuity.—The amount of an ordinary annuity is the sum at maturity of all the periodic payments, plus the interest on all payments.

Analysis of compound interest.—To be able to understand annuities, it is necessary to have a knowledge of how compound interest tables are built up. It was shown under the subject of Compound Interest that $(1 + i)^n$, or $(1.06)^4$, for example, equals the amount found in the 6 per cent column of a compound interest table, the fourth number from the top. An analysis shows that this amount is composed of three things: a principal of 1, an annual addition of simple interest of .06, and "interest on interest" of .02247696.

A further analysis might be made as follows:

	Jan. 1, 1921	*Dec. 31, 1921*	*Dec. 31, 1922*	*Dec. 31, 1923*	*Dec. 31, 1924*	*Total*
Invested Jan. 1, 1921	1.					1.
1st annual interest on 1.		.06				
Interest on .06			.0036	.0036	.0036	
" " 1st .0036				.000216	.000216	
" " 2nd .0036					.000216	
" " .000216					.00001296	.07146096
2nd annual interest on 1.			.06			
Interest on .06				.0036	.0036	
" " .0036					.000216	.067416
3rd annual interest on 1.				.06		
Interest on .06					.0036	.0636
4th annual interest on 1.					.06	.06
Total						1.26247696

Summarizing the above, three distinct parts appear:

1.	The principal	1.00
2.	The four *equal annual amounts* of simple interest of $.06	.24
3.	The accumulations of "interest on interest"	.02247696
	Total	1.26247696

The above may again be shown as:

1.	Principal	$1.00
2.	Compound interest	.26247696
3.	Compound amount	$1.26247696

Relation of compound interest and annuities.—In the above table of analysis, it can be seen that there are a series of payments of .06 each at regular stated intervals of one year, and that compound interest is calculated on each of these .06 payments until the end of the fourth year.

Hence, .26247696 is the amount of an annuity of .06 per year for 4 years at 6%.
$.26247696 \div 6 = .04374616$, or the amount of an annuity of .01 for 4 years at 6%.
$.04374616 \times 1.00 = 4.374616$, the amount of an annuity of 1 for 4 years at 6%.

Amount of annuities found from a compound interest table.—The above discussion was limited to the finding of the

amount of an annuity at 6% for four years. The same process can be used for any number of periods at any interest rate.

Procedure.—(*a*) Calculate the compound interest on 1 for the number of periods at the given rate.

(*b*) Divide the compound interest on 1 by the rate per cent expressed decimally to get the amount of the annuity of 1.

(*c*) Multiply the amount of the annuity of 1 by the number of dollars of each annuity rent.

Example.—Assume that it is desired to find the amount of an annuity of $100 for five years at 6%.

Solution.

1.338226 = compound amount of 1 at 6%, or $(1.06)^5$.
1.338226 − 1 = .338226, compound interest of 1 at 6%.
.338226 ÷ .06 = 5.6371, amount of annuity of 1.
$100 × 5.6371 = $563.71, amount of annuity of $100 for 5 years at 6%.

From the above, the following may be derived:

$(1+i)^n$	or	$(1.06)^5$	=	1.338226, compound amount.
$(1+i)^n - 1$	or	$(1.06)^5 - 1$	=	.338226, compound interest.
$\frac{(1+i)^n - 1}{i}$	or	$\frac{(1.06)^5 - 1}{.06}$	=	5.6371, amount of an annuity of 1.
$P\left(\frac{(1+i)^n - 1}{i}\right)$	or	$100\left(\frac{(1.06)^5 - 1}{.06}\right)$	=	$563.71, amount of an annuity of $100.

Verification.

Assumed Dates			
Jan. 1, 1920	Contract made.		
Dec. 31, 1920	Rent paid		$100.00
Dec. 31, 1921	Interest at 6% on $100	$ 6.00	
	Rent paid	100.00	106.00
Jan. 1, 1922	Amount of annuity for 2 years		$206.00
Dec. 31, 1922	Interest at 6% on $206	$ 12.36	
	Rent paid	100.00	$112.36
Jan. 1, 1923	Amount of annuity for 3 years		$318.36
Dec. 31, 1923	Interest at 6% on $318.36	19.10	
	Rent paid	100.00	119.10
Jan. 1, 1924	Amount of annuity for 4 years		$437.46
Dec. 31, 1924	Interest at 6% on $437.46	26.25	
	Rent paid	100.00	126.25
	Total		$563.71

Formation of table of amounts of annuities.—A table of amounts of an annuity may be built up from a table of compound amounts of 1, thus:

TABLE

1	*2*	*3*	*4*
End of Period	*Compound Interest Table*	*Compound Interest Accumulation*	*Annuity Table of Amounts*
1	$1.06	$.06	$1.00
2	1.1236	.1236	2.06
3	1.191016	.191016	3.1836
4	1.262477	.262477	4.374616
5	1.338226	.338226	5.637093

Column 4 is found by dividing Column 3 by the rate per cent, which in this case is 6.

Semiannual or quarterly basis.—If the rents are payable each six months, or each three months, and the interest is to be compounded on the same dates, the method of using the rate per period and the time in periods is preferable to the method of finding the effective interest for a year and using the time in years. Only the one method will be illustrated.

Procedure.—(*a*) Find the nominal rate per period.

(*b*) Find the number of periods.

(*c*) Determine the compound interest on 1 for the number of periods at the nominal rate per period.

(*d*) Divide the compound interest found in (*c*) by the nominal rate per cent per period.

(*e*) Multiply the amount of the annuity of 1 by the number of dollars of each periodic rent.

Example.—What will be the amount of an ordinary annuity of 8 rents of $50 each, payable each six months, interest at 6% per year, compounded semiannually?

Algebraic Formula.

$$R\left(\frac{(1+i)^n-1}{i}\right) = \text{Amount of annuity.}$$

Arithmetical Substitution.

$$50\left(\frac{(1.03)^8-1}{.03}\right) = \text{Amount of annuity.}$$

Solution.

$.06 \div 2$	=	.03,	rate per cent per period.
4×2	=	8,	number of periods.
$(1.03)^8$	=	1.26677,	1 at compound interest for 8 periods.
$1.26677 - 1$	=	.26677,	compound interest on 1 for 8 periods.
$.26677 \div .03$	=	8.8923,	amount of annuity of 1 for 8 periods.
$\$50 \times 8.8923$	=	$444.62,	amount of annuity of $50 for 8 periods.

Problems.

1. Find the amount of an ordinary annuity of

(a)	$1	at	6%	for	5	years,	rents	and	interest	payable	annually.
(b)	$1	"	8%	"	8	"	"	"	"	"	"
(c)	$1	"	4%	"	10	"	"	"	"	"	semiannually.
(d)	$250	"	6%	"	16	"	"	"	"	"	"
(e)	$50	"	6%	"	50	"	"	"	"	"	annually.
(f)	$100	"	4%	"	40	"	"	"	"	"	"

2. Construct a table of the amounts of annuities for ten years at 4%, using columns headed "Compound Interest Table," "Compound Interest," and "Amount of Annuities," as illustrated on page 230.

3. Construct a table of analysis of the amount of an annuity of 1 for 5 years at 5%.

Rent of an ordinary annuity.—The periodic payment made to or withdrawn from an annuity fund is called the "rent of the annuity." It often happens that a company wishes to know the periodic payments, which, with interest, will in a definite future time equal a required amount.

Procedure.—(*a*) Compute the amount of the annuity of 1 for the given number of periods at the given rate per period.

(*b*) Divide the number of dollars of the required amount by the amount of an annuity of 1. The result will be the rent or periodic payment.

Example.—What should be the amount of each equal annual payment into a fund, which, in four years at 6%, interest compounded annually, will amount to $1000?

Algebraic Formula.

$$\frac{\frac{P}{(1+i)^n - 1}}{i} = \text{Rent.}$$

Arithmetical Substitution.

$$\frac{\frac{1000}{(1.06)^4 - 1}}{.06} = \text{Rent.}$$

Solution.

$(1.06)^4 = 1.262477$, compound amount of 1 for 4 years at 6%.
$1.262477 - 1 = .262477$, compound interest on 1 for 4 years at 6%.
$.262477 \div .06 = 4.374616$, amount of annuity of 1 for 4 years at 6%.
$\$1000 \div 4.374616 = \$228.59$, rent required to accumulate \$1000 at the end of 4 years.

Verification.

Assumed Dates		
Dec. 31, 1920	Rent at end of first year	\$ 228.59
Dec. 31, 1921	Interest on \$228.59	13.72
Dec. 31, 1921	Rent at end of second year	228.59
	Amount of annuity at the end of 2 years	\$ 470.90
Dec. 31, 1922	Interest on \$470.90	28.25
Dec. 31, 1922	Rent at end of third year	228.59
	Amount of annuity at the end of 3 years	\$ 727.74
Dec. 31, 1923	Interest on \$727.74	43.66
Dec. 31, 1923	Rent at end of fourth year	228.60
	Amount of annuity at the end of 4 years at 6%	\$1000.00

TABLE

1 *End of Period*	*2* *Rent*	*3* *Interest Accumulation*	*4* *Total Amount*
1	\$228.59	....	\$ 228.59
2	228.59	\$13.72	470.90
3	228.59	28.25	727.74
4	228.60	43.66	1000.00

Problems.

1. How much must be placed in the savings bank at the end of each quarter for 4 years at 4% compound interest to accumulate to \$1000, at the date of the last deposit? Construct formula, solution, and verification.

2. A company owes \$800, due in three years. How much must be set aside semiannually at 6%, so that the payments and interest will accumulate enough to pay the debt at maturity? Construct formula, solution, and table.

3. A company issued bonds for \$10,000 due in ten years, and desires to know how much must be set aside each three months, money being worth 8%, interest com-

pounded quarterly, to meet the payments on the bonds when due.

4. In auditing the books of a corporation you find that, in order to provide a sum to redeem a mortgage of $100,000, falling due at the end of ten years, a reserve of $8000 per annum has been set aside annually for three years, but that, contrary to intention, the company has failed to accumulate interest thereon. Assuming interest at 4% (convertible annually), what should have been the total accumulations to date, and what amount should now be set aside annually for the next seven years in order to complete the sinking fund? $(1.04)^7 = 1.31593$.—*American Institute Examination.*

Note.—The above problem has different interpretations. The meaning is very uncertain.

(*a*) Does it mean that a reserve only has been set aside, but no fund to cover the reserve?

(*b*) Does it mean that $8000 has been set aside for three years and has not accumulated interest?

(*c*) Does the latter part of the problem mean that the calculations are to be made for the $8000 a year for three years, and the balance is to be set aside in the seven years remaining?

Use the last interpretation for the solution of this problem.

Effective interest in annuities.—Very often the rents are paid on an annual basis and the interest is compounded on a semiannual or quarterly basis; if such is the case, it is necessary to use the effective rate of interest.

If the interest is compounded more or less frequently than the rents are paid, it is necessary to convert the nominal interest rate to an effective interest rate applicable to the rent periods.

Procedure.—(*a*) Calculate the effective periodic interest on the basis of the rent periods.

(*b*) Calculate the amount of an annuity of 1, using the effective rate per period, the number of periods corresponding to the number of rents.

(*c*) Multiply the amount of an annuity of 1 found in (*b*) by the number of dollars of each rent.

Example.—What will be the amount of an annuity, the annual payments of which are $100 for four years at 6%, interest compounded quarterly?

The solution to this problem is stated in two parts:

1. Calculation of the effective rate per year.
2. Calculation of the amount of the annuity of $100.

Part 1.

Algebraic Formula.

$$\left(1+\frac{j}{m}\right)^m - 1 = \text{Effective rate.}$$

Arithmetical Substitution.

$$\left(1+\frac{.06}{4}\right)^4 - 1 = \text{Effective rate.}$$

Solution.

$.06 \div 4$	$= .015,$	rate per period.
$1. + .015$	$= 1.015,$	ratio of increase.
$(1.015)^4$	$= 1.0613635,$	compound amount of 1 for 1 year.
$1.0613635 - 1$	$= .0613635,$	effective rate per year.

Part 2.

Use the "effective rate" found in part 1, and proceed as in an ordinary annuity.

Algebraic Formula.

$$R\left[\frac{(1+i)^n - 1}{i}\right] = \text{Amount.}$$

Arithmetical Substitution.

$$100\left[\frac{(1.0613635)^4 - 1}{.0613635}\right] = \text{Amount of annuity.}$$

The solution would then be:

$(1.0613635)^4$	$=$ 1.2689855,	compound amount of 1 for four periods at 6.13635%.
$1.2689855 - 1.$	$=$ .2689855,	compound interest on 1 for four periods at 6.13635%.
$.2689855 \div .0613635$	$=$ 4.383477,	amount of annuity of 1 for four periods at 6.13635%.
$\$100 \times 4.383477$	$= \$438.35,$	amount of annuity of $100 for 4 years.

Verification.

First year:			
Rent at end of year	=		$100.00
Second year:			
$100 × .0613635 (effective rate)	=	$ 6.136	
Rent	=	100.00	106.136
Amount of annuity for 2 years	=		206.136
Third year:			
$206.136 × .0613635	=	12.649	
Rent	=	100.00	112.649
Amount of annuity for 3 years	=		318.785
Fourth year:			
$318.785 × .0613635	=	19.561	
Rent	=	100.00	119.561
Amount of annuity for 4 years	=		$438.35

Problems.

1. What is the amount of a ten-year annuity if the payments are $500 each and are to be made at the end of each year? Interest is at 4%, compounded quarterly.

2. John Eastman desires to know how much he will have in the savings bank at the end of twenty years if he deposits $100 at the end of each six months. The bank pays 4%, interest compounded at the end of each quarterly period.

3. $550 is put in the bank at the end of each year for ten years. If the bank credits the account quarterly with interest at 5% (nominal rate), what will be the accumulation? Establish a proof.

Sinking fund contributions.—A sinking fund produced by equal periodical payments accumulating at compound interest is an annuity. The principles applicable to annuities will be further illustrated with special reference to sinking funds.

Procedure.—(*a*) Divide the number of dollars required in the total fund by the amount of the annuity of 1 for the required number of periods at the given rate.

Example.—A company borrowed $2500 for five years and established a sinking fund to provide for the payment of the debt, the contributions to be made at the end of each year. Money being worth 6%, what should be the annual contribution?

Algebraic Formula.

$$\frac{P}{\frac{(1+i)^n-1}{i}} = \text{Sinking fund contribution.}$$

Arithmetical Substitution.

$$\frac{\$2500}{\frac{(1.06)^5-1}{.06}} = \text{Sinking fund contribution.}$$

Solution.

$(1.06)^5 = 1.338225$, compound amount of 1 for 5 periods at 6%.
$1.338225 - 1 = .338225$, compound interest on 1 for 5 periods at 6%.
$.338225 \div .06 = 5.63709$, amount of ordinary annuity of 1 for 5 periods at 6%.
$\$2500 \div 5.63709 = \$443.49$, contribution to sinking fund.

TABLE OF SINKING FUND CONTRIBUTIONS

1 *End of Period*	*2* *Contribution*	*3* *Interest*	*4* *Yearly Increase*	*5* *Total Fund*
1	$ 443.49	.....	$ 443.49	$ 443.49
2	443.49	$ 26.61	470.10	913.59
3	443.49	54.82	498.31	1411.90
4	443.49	84.71	528.20	1940.10
5	443.49	116.41	559.90	2500.00
	2217.45	282.55	2500.00	

Problems.

1. What sum must be set aside at the end of each six months so that the semiannual payments and interest accrued thereon will amount to $5000 in four years? Interest at 4% is to be compounded semiannually. Construct table as above.

2. What amounts set aside semiannually for four years at 5% interest, compounded semiannually, will produce enough to cancel a debt of $60,000 due in 4 years? Construct table as above.

3. You are called upon to state what is the annual sinking fund necessary to redeem a principal sum of $1,000,000 due thirty years hence, it being assumed that the annual sums set aside are invested at compound interest at 5%. State

what computations you would make to arrive at the result desired. You need not work out the computations.—*American Institute Examination.*

Note.—The question probably means, what method would you use in arriving at the value of $(1.05)^{30}$, as well as the method of finding the rent.

While the problem states, "You need not work," it is desirable to answer the questions called for, and also to construct a table for the first six years.

4. A company has acquired machinery which cost $100,000, which it expects to be able to use for ten years; the scrap value at the end of that time is estimated at $25,000. A bond issue of $75,000 due in ten years, bearing 6% interest, and secured by a mortgage on the machinery, was floated at 98 soon after the purchase of the machinery. The trust indenture requires that at the end of each year, before the payment of dividends, a sum shall be set aside and charged against earnings sufficient to provide a sinking fund on a 5% basis for the redemption of the bonds at maturity.

The president of the company is in favor of providing a reserve for depreciation on the machinery by the sinking fund method, using 5% as a basis, although he does not advocate creating a replacement fund for the machinery as well as a sinking fund for the bonds. The treasurer contends that it is not necessary to provide any reserve for depreciation, asserting as his reason that the creation of a sinking fund reserve and a reserve for depreciation would involve a double charge against profits, and further, that as the sinking fund reserve is obligatory, the depreciation reserve is not required.

(*a*) Compute the amount of the annual contribution to the sinking fund for the redemption of the bonds.

(*b*) Set up a table showing the accumulation of the fund, on the assumption that it earned exactly 5%. Also indicate the annual entries for the sinking fund and for the sinking fund reserve.

(*c*) Give your opinion as to whether or not there should be a reserve for depreciation as well as a sinking fund reserve. If the sinking fund reserve is sufficient, what disposition will eventually be made of it? If two reserves are

necessary, when and in what manner will they be closed out? —*American Institute Examination.*

5. The annual sinking fund has been fixed at $83,290.94 to redeem a bond issue of $1,000,000 at the end of ten years. It is argued against this plan that the payments should be $100,000 per annum. Demonstrate which of these two amounts is the correct one, assuming the fund to be invested meantime at 4% interest, compounded annually. Show also the amount of the saving effected by the smaller sinking fund, discounted to maturity, and the amount to be returned to the borrower at the end of the period if the larger sinking fund ($100,000) were adopted. $(1.04)^{10} = 1.480244$.—*American Institute Examination.*

6. The annual sinking-fund deposit, with interest at 6% per annum, necessary to accumulate $500,000 over a period of 80 years is $286.27. The amount of an annuity of one dollar at 6% at the end of various periods is as follows: 20 years, $36.78559120; 40 years, $154.76196562; 60 years, $533.12818089; 80 years, $1,746.59989137. Show, by means of a table, how a sinking-fund-method reserve for amortization of a franchise, which cost $500,000 and has 80 years to run, would compare with a reserve for the same purpose derived by the straight-line method.—*American Institute Examination.*

Present value of an ordinary annuity.—The present value of an ordinary annuity is the sum which, if put at compound interest, will produce the periodic rents of an annuity contract when they become due.

The present value of each periodic rent may be found separately; the present values of these separate periodic rents may be added, and the sum will be the present worth of the annuity.

Illustration 1.

Assume that it is desired to find the value on January 1, 1924, of a series of four annual periodic payments of six cents each. Payments are to be made at the end of each year.

TABLE OF ANALYSIS OF ILLUSTRATION 1

Contract Entered Into Jan. 1, 1924		Payments To Be Made On Dec. 31, 1924	Dec. 31, 1925	Dec. 31, 1926	Dec. 31, 1927
Present Value of:					
1st rent	$.056603 = $\frac{1}{(1.06)}$, or .94339 ×	$.06			
2nd rent	.053399 = $\frac{1}{(1.06)^2}$, or .88999 ×		$.06		
3rd rent	.050377 = $\frac{1}{(1.06)^3}$, or .83962 ×			$.06	
4th rent	.047525 = $\frac{1}{(1.06)^4}$, or .79209 ×				$.06
Total	$.207904 = Present value of an annuity of $.06.				

Computation of the compound discount on 1 for four periods at the rate of 6% is as follows:

$(1.06)^4$ = 1.262477, compound amount of 1 for 4 periods at 6%.
1. ÷ 1.262477 = .792094, present value of 1 for 4 periods at 6%.
1. — .792094 = $.207904, compound discount on 1 for 4 periods at 6%.

It will be noted that the present value at 6% of an annuity of 4 rents of 6 cents each, as computed above, is the same as the compound discount on $1.00 for 4 periods at 6%. Substituting symbols for figures, it is apparent that the compound discount on 1 for n periods at the rate of i is equal to the present value of an annuity for n periods, the rents of which, when stated in cents, are the same as i.

In the illustration and explanation just given, the basis of calculation is the periodic payment of 6 cents, and is stated as "an annuity of 6 cents." The basis most used, or the common basis, is $1.00 and is expressed as an "annuity of 1." (Annuity tables are built on this basis.) Therefore, in order to find the present value of an annuity of 1, divide the compound discount by the rate per cent expressed decimally. Using the figures given above, the calculation would be .207904 ÷ .06 = 3.465105, or the annuity of 1 for 4 periods at 6%.

Procedure.—(*a*) Calculate the compound discount on 1 for the required number of periods and at the required rate per cent.

(*b*) Divide the compound discount by the given rate per cent, expressed decimally. The quotient will be the present value of an annuity of 1.

(*c*) Multiply the present value of the annuity of 1 by the number of dollars of each rent.

Example.—What is the present worth of an annuity contract at the beginning of the first year by which a payment of $100 is to be made at the end of each year for four years, money being worth 6%, compounded annually?

Algebraic Formula.

$$R\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right] = \text{Present worth of annuity.}$$

Arithmetical Substitution.

$$100\left[\frac{1-\frac{1}{(1.06)^4}}{.06}\right] = \text{Present worth of annuity.}$$

Solution.

$(1.06)^4$	=	1.262477, compound amount of 1 for 4 years at 6%.
$1 \div 1.262477$	=	.792094, present worth of 1 for 4 years at 6%.
$1 - .792094$	=	.207906, compound discount on 1 for 4 years at 6%.
$.207906 \div .06$	=	3.4651, present worth of an annuity of 1 for 4 years at 6%.
$3.4651 \times \$100$	=	$346.51, present worth of an annuity of $100.

Verification.

Assumed Dates			
Jan. 1, 1919	Present worth of contract.....		$346.51
	Deduct:		
Jan. 1, 1920	Rent	$100.00	
	Interest on $346.51 at 6%.....	20.79	
	Reduction in value of annuity contract		79.21
	Present worth		$267.30
	Deduct:		
Jan. 1, 1921	Rent	$100.00	
	Interest on $267.30.....	16.04	
	Reduction in value of annuity contract		83.96
	Present worth		$183.34
	Deduct:		
Jan. 1, 1922	Rent	$100.00	
	Interest on $183.34.....	11.00	
	Reduction in value of annuity contract		89.00
	Present worth		$ 94.34
	Deduct:		
Jan. 1, 1923	Rent	$100.00	
	Interest on $94.34.....	5.66	
	Reduction in value of annuity contract		94.34
			$ .00

Amortization.—Payments made on the principal, as shown in the above example, are known as "amortization payments." Amortization is the gradual repayment of the principal through the two opposing forces of compound interest, and periodic payments. Compound interest increases the value, while the payments reduce it. In the "verification" of present worth of an annuity, it can be seen that the difference between the two opposing forces—the compound interest and the payment—is the amount by which the principal is reduced. The amount of this reduction is the *amortization.*

Problems.

1. Find the present worth of an ordinary annuity of:

(a) \$100 annual payments, at 4%, for 6 years.
(b) \$25 annual payments, at 5%, for 10 years.
(c) \$50 annual payments, at 3%, for 20 years.
(d) \$5 semiannual payments, at 6%, for 3 years.
(e) \$40 semiannual payments, at 6%, for 3 years.
(f) Submit solution and verification showing the applications of the payments as to interest, amortization of principal, and new principal each year for the present worth of an annuity contract by which \$200 is to be paid each six months for three years, money being worth 8%, compounded semiannually.

2. An annuity of \$200, payable at the end of each year for ten years, is worth how much at the beginning of the first period, if money is worth 5%, compounded semiannually? Submit columnar table.

3. *A* owns an annuity of \$50 per annum, the first payment of which falls due one year hence, and which continues for a period of twenty years certain. State:

(a) The present value of the benefit.
(b) The amount which he will have accumulated at the end of the period if he invests each annuity payment as it becomes due. Assume interest at 4%, payable annually. In this connection, the value of $(1.04)^{20}$ is stated to be equal to 2.191123.—*American Institute Examination.*

4. A man purchases a house for $2500 cash, and sixteen notes of $500 each without interest, due one at the end of each six months until paid. If money is worth 4%, compounded semiannually, what is the cash value of the property?

5. What is the cash value of a contract which calls for the payment of $50 at the end of each month for 8⅓ years, if money is worth 6%, convertible monthly?

Rents or periodic payments of the present value of an ordinary annuity.—If the present value of the annuity, the rate per cent, and the time are given, the rents or periodic payments may be calculated as follows:

Procedure.—(*a*) Determine the present value of an annuity of 1 for the required number of periods, at the rate per period.

(*b*) Divide the given present value of the annuity by the present value of the annuity of 1 found in (*a*).

Example.—What annual rent will be produced by an ordinary annuity, the present value of which is $346.51, if there are four rents, and money is worth 6%?

Algebraic Formula.

$$\frac{P}{\dfrac{1-\dfrac{1}{(1+i)^n}}{i}} = \text{Rent of present worth of annuity.}$$

Arithmetical Substitution.

$$\frac{346.51}{\dfrac{1-\dfrac{1}{(1.06)^4}}{.06}} = \text{Rent of present worth of annuity.}$$

Solution.

$(1.06)^4$	=	1.262477, compound amount of 1 for 4 years at 6%.
$1 \div 1.262477$	=	.792094, present worth of 1 for 4 years at 6%.
$1 - .792094$	=	.207906, compound discount on 1 for 4 years at 6%.
$.207906 \div .06$	=	3.4651, present worth of an annuity of 1 for 4 years at 6%.
$\$346.51 \div 3.4651 = \$100,$		rent of annuity.

See verification shown under Present Worth of Annuity.

Problems.

1. If the present value of an annuity contract is $6000, what amount must be paid at the end of each year for six years, to cancel the obligation, money being worth 4%, interest compounded annually?

2. An annuity contract is worth $15,000 at the present time. If the time to run is eight years, and money is worth 5%, compounded semiannually, what periodic payment should be paid at the end of each six months to cancel the contract?

3. The present value of a ten-year annuity contract is $4000. If money is worth 4%, convertible quarterly, what amount should be paid at the end of each year to cancel the contract?

Payment of debts by installments.—In many contracts, the agreement states that the principal of the debt and the interest are to be paid in equal periodic payments; each payment is to cancel the interest due to date, and the balance is to be applied toward the repayment of the principal.

The expression "payment of debt by installment" is confusing to many. But, owing to the vast number of contracts based on installment payments which are being made daily, both in real estate and merchandise, it is important to be able to apply this principle.

The procedure, formula, and solution should be similar to those given under "Rents or periodic payments of the present value of an ordinary annuity."

Example.—If Smith borrows of Jones $1000 at 6%, and agrees to cancel the debt (principal and interest) in five equal annual payments, what will be the size of each payment?

Formula.—The formula given under "Rent of Present Worth of Annuity" is applicable here.

TABLE

1 *End of Period*	*2* *Payment Made*	*3* *To Cancel Interest*	*4* *Amortization of Principal*	*5* *Balance of Principal*
				$1000.00
1	$ 237.39	$ 60.00	$177.39	822.61
2	237.39	49.36	188.03	634.58
3	237.39	38.07	199.32	435.26
4	237.39	26.11	211.28	223.98
5	237.39	13.44	223.95	.03
Totals	$1186.95	$186.98	$999.97	

A slight error, as in the above solution, is likely to occur in many such problems and is due to not carrying the computations to more decimal places. In such cases the difference should be corrected in the last payment. In the table,

the fifth payment would be $237.42, instead of $237.39. This would leave column five with no remainder, and column four would have a total of $1000.

Problems.

1. What amount set aside at the end of each year for six years at 5% will exactly pay (principal and interest) a loan of $3500?

2. *A* borrowed $5000 of *B* for four years and agreed to pay the principal and interest in eight equal semiannual payments to be made at the end of each period, interest being at the rate of 8%. Solve for size of payments and prove by table.

3. A corporation wants to retire a debt of $105,000, bearing 5% interest, payable annually. The tenth payment, including interest, is to be $15,000. The other nine periodical payments are all to include interest and to be of the same amount.

The amount of each of such nine payments is required. $(1.05)^9 = 1.551328$.—*American Institute Examination.*

4. *A* contracts for the purchase of a house, agreeing to pay for it in equal annual installments over a period of ten years. The cash value of the house is $6000. If money is worth 6%, convertible quarterly, how much should each payment be, if made at the end of each period?

5. (*a*) What terminable annuity, payable quarterly, for ten years, would be required to repay a loan of $32,840, the nominal rate of interest being 4% per annum?

(*b*) What would be the required amount to set aside annually providing for a sinking fund to repay loan, as above, at the end of ten years instead of repaying it quarterly? Given $(1.01)^{40} = 1.4889$.—*American Institute Examination.*

Term of annuities.—When the rate per cent, the amount of annuity, and the size of each periodic payment are stated, it is possible to calculate the number of periodic payments to be made.

To calculate the time, it is necessary to resort to equations and logarithms.

Procedure.—(*a*) Divide the amount of the annuity by the number of dollars in one of the periodic payments to find the amount of the annuity of 1 at the given rate.

(*b*) Multiply the amount of the annuity of 1 by the rate per period expressed decimally, to find the compound interest on 1 for the unknown time.

(*c*) Add 1 to the compound interest on 1 to find the compound amount of 1.

(*d*) Determine the log of the compound amount of 1 as found in (*c*).

(*e*) Determine the log of 1 plus the rate per cent for the period expressed decimally, that is, the ratio of increase.

(*f*) Divide the log of the compound amount of 1, (*d*), by the log of the ratio of increase (*e*), to find the number of periods, or periodic payments.

Example.—Find the term of an annuity, if the amount is \$1318.08, the annual rent is \$100 each year, and the rate is 6%.

Algebraic Formula.

$$\frac{\text{Log}\left[1+\left(\frac{\text{Amount of annuity}}{\text{Rent}}\times\text{rate}\right)\right]}{\text{Log (Ratio of increase)}}=\text{Term.}$$

Arithmetical Substitution.

$$\frac{\text{Log}\left[1+\left(\frac{1318.08}{100.}\times .06\right)\right]}{\text{Log (1.06)}}=\text{Term.}$$

Solution.

Dividing by rent,	1318.08 ÷ 100	= 13.1808	(1)
Multiplying by rate,	13.1808 × .06	= .790848	(2)
Adding to 1,	1. + .790848	= 1.790848	(3)
Log	1.790848	= 0.253059	(4)
Log	1.06	= 0.025306	(5)
Dividing,	0.253059 ÷ 0.025306	= 10	(6)

As the rents are to be paid annually, the annuity term will be for ten years.

Illustration 2.

If the payments are to be made more often than once each year, the rate should be reduced to a rate per period.

Example.—If a man purchases a house for $2629.02 and agrees to pay $500 down and $50 at the end of each month, how many months will it take to pay the debt if each payment is to cancel the interest to date, and the balance is to apply on the principal? The debt bears 6% interest.

Reducing the rate to a rate per period, $.06 \div 12 = .005$.

Algebraic Formula.

$$\frac{\text{Log}\left[\frac{1}{1-\left(\frac{\text{Debt}}{\text{Rent}} \times i\right)}\right]}{\text{Log}\quad 1+i} = \text{Term.}$$

Arithmetical Substitution.

$$\frac{\text{Log}\left[\frac{1}{1-\left(\frac{2129.02}{50} \times .005\right)}\right]}{\text{Log}\quad 1.005} = \text{Term.}$$

Solution.

Dividing,	$2129.02 \div 50$	$= 42.5804$		(1)
Multiplying,	$42.5804 \times .005$	$= .212902$		(2)
Subtracting,	$1 - .212902$	$= .787098$		(3)
Dividing,	$1 \div .787098$	$= 1.270489$		(4)
Log of	1.270489	$= 0.1039709$		(5)
Log of	1.005	$= 0.0021661$		(6)
Dividing	$0.1039709 \div 0.0021661$	$= 48,$	approximately	(7)

or 48 monthly payments.

Problems.

1. Enos Anderson purchased a house and lot for $6800. He agreed to pay $2800 cash and $50 at the end of each month until paid. How many months will it take to pay the debt, if each payment is to cancel the interest due and the balance is to apply on the principal? The interest rate is 6%.

2. *A* sells his residence for $7500. He receives a down payment of $1500. The balance is to be paid on a monthly plan of $75 each month until paid. The monthly payment is to cancel the interest first, and the balance is to apply on the debt. The contract states interest is at the rate of 4% per annum. How long will it take the buyer to pay off the debt?

3. Jones buys a farm for $15,000 and agrees to pay $5000 down, the balance to draw interest at 5% until paid. Any payments made are to apply on interest due to date and if the payments exceed the interest due, the balance will be applied on the reduction of the principal. He desires to know how long it will take to pay for the farm if he can make equal quarterly payments of $300.

Use of effective rate in annuities.—If the periods of the compounding of interest occur more often than the periods for the paying of the rents, or vice versa, it is necessary to reduce the rate of interest to an effective rate for a period corresponding to the periods of the payments of the rents.

Procedure.—(*a*) Calculate the effective rate of interest for one rent period.

(*b*)—*1.* If the amount of annuity is known, use the procedure given under Illustration 1 on page 245.

(*b*)—*2.* If the present value of the annuity is given, use the procedure given under Illustration 2 on page 246.

Example.—The rents of $100 each are to be paid annually, the amount is $318.64, and the interest rate is 6%, compounded semiannually. Find the term.

Part I.

Algebraic Formula.

$$\left(1+\frac{j}{m}\right)^{m}-1=\text{Effective rate.}$$

Arithmetical Substitution.

$$\left(1+\frac{.06}{2}\right)^{2}-1=\text{Effective rate.}$$

Solution.

$$\left(1+\frac{.06}{2}\right)^{2}=1.0609,\ \text{effective ratio of increase for one year.}$$

$$1.0609-1=.0609,\ \text{effective rate.}$$

Part II.

As the problem states the amount of an annuity, it will be solved by procedure (*b*)—*1.*

Algebraic Formula.

$$\frac{\text{Log}\left[1+\left(\frac{\text{Amount}}{\text{Rent}}\times i\right)\right]}{\text{Log}\quad\text{Ratio of increase}}=\text{Term.}$$

Arithmetical Substitution.

$$\frac{\text{Log}\left[1+\left(\frac{318.64}{100}\times .0609\right)\right]}{\text{Log}\qquad 1.0609} = \text{Term.}$$

Solution.

Dividing,	$316.14 \div 100$	$= 3.1614$	(1)
Multiplying,	$3.1614 \times .0609$	$= .1940517$	(2)
Adding,	$1. + .1940517$	$= 1.1940517$	(3)
Log of	1.1940517	$= 0.077022$	(4)
Log of	1.0609	$= 0.025674$	(5)
Dividing,	Log 0.077022 $\div$ Log 0.025674	$= 3$	(6)

The result, 3, indicates that there are three annual payments of $100 each.

Verification.

First year:			
Rent	=		$100.00
Second year:			
$100 × .0609	=	$ 6.09	
Rent	=	100.00	106.09
			$206.09
Third year:			
$206.09 × .0609	=	12.55	
Rent	=	100.00	112.55
Present value	=		$318.64

Problems.

1. What is the number of periods in each of the following?

	Amount of an Ordinary Annuity	*Rents*	*Rate*
(a)	$ 8.0191	$ 1.	4½%
(b)	663.29	100.	4%
(c)	9549.11	1000.	5%
(d)	1061.82	200.	3%

2. A man desired to repay a debt of $5000 by paying $200 at the end of each three months. If money is worth 6%, convertible semiannually, how many payments will he have to make?

3. If payments of $1000 are made annually and money is worth 6%, convertible quarterly, how many full periodic

payments must be made to produce $10,520, if payments are made at the end of each period?

4. John Adams agrees to make payments at the end of each month of $10 each to repay a loan of $1000, and interest at 6% convertible monthly. How long will it take to cancel the debt?

5. A building is purchased for $15,000, to be paid for in semiannual installments of $1000 each, and interest is 6%, compounded semiannually. How many payments will be required, if made at the end of each 6 months' period?

Rate of annuity.—The mathematical theory of an annuity deals with six elements—term, rent, rate of interest, number of periodic payments, present value of an annuity, and amount of an annuity. If the rate and any other four elements are given, the missing elements may be found, but as the rate is used twice in annuity calculations, it can be only approximated.

To get an approximate rate.—By inspection, find a trial rate and test it to see if the estimated rate used with the given component parts will produce an amount equivalent or nearly equivalent to the amount of the annuity. If the estimated rate proves to be near the required rate, use it as a basis, and select another rate so that of the two chosen rates one will be more and the other less than the required rate. Proceed to select the approximate rate by interpolation.

The test of the chosen rates may be made by means of an annuity table, or by the calculation of the amount of an annuity of 1.

Selection by use of an annuity table.—

Procedure.—(*a*) Divide the given amount of the annuity by the annuity rent to find the amount of an annuity of 1.

(*b*) Choose from the annuity table the two amounts nearest to the amount found in (*a*).

(*c*) Proceed by interpolation to find the required rate.

Example.—What is the rate for an ordinary annuity having five annual rents of $200 each, the amount of which will be $1099.62?

SECTION OF ANNUITY TABLE

Periods	*3½%*	*4%*	*4½%*	*5%*	*6%*
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.03500	2.04000	2.04500	2.05000	2.06000
3	3.10622	3.12160	3.13702	3.15250	3.18360
4	4.21494	4.24646	4.27819	4.31012	4.37461
5	5.36246	5.41632	5.47071	5.52563	5.63709

Solution.

$1099.62 ÷ $200 = $5.4981, amount of an annuity of $1 for five periods.

Interpolation.—The second step is to choose from the annuity table, horizontally to the right of the fifth period, the two amounts nearest to $5.4981. By inspection, the amount, $5.47071, found in the 4½% column is the nearest one under, and $5.52563, found in the 5% column, is the nearest one over $5.4981. Using these rates and amounts as a basis, the following may be derived:

Amount of a $1 annuity at 5%	=	$5.52563
Amount of a $1 annuity at 4½%	=	5.47071
Difference in amount caused by ½% difference in interest rate	=	.05492
Amount of a $1 annuity at unknown rate	=	$5.4981
Less amount at 4½%	=	5.47071
Difference	=	.02739

$\frac{.02739}{.05492}$ of ½% = .2493 +, or approximately ¼%.

4½% + ¼% = 4¾%, the required rate.

Selection by calculation of amount of annuities.—If no annuity table is at hand, it is necessary to obtain the two basic rates by the calculation of the amounts of annuities, using estimated rates. The procedure might be—

First.—As this is a five payment annuity and each payment is for $200, the total paid in would be $1000. Deducting this $1000 from the amount, $1099.62, the interest is found to be $99.62.

Second.—The payments would draw interest thus:

$200	for	4 years
200	"	3 "
200	"	2 "
200	"	1 year
200	"	0 years
		10 years

Thus we have an average of $200 for ten years.

By calculating simple interest on $200 for ten years at 5%, the amount of interest is found to be $100, or a little more than the interest in the problem, disregarding the compounding. Thus, it can be seen that the interest is probably less than 5%, but as the compounding of interest is infrequent, the variation will be small. Hence, it is advisable to try the rate of 5%.

First trial rate.—A trial rate of 5% will be used. The solution by the 5% rate is as follows:

$$200\left(\frac{(1.05)^5-1}{.05}\right) = \text{Amount of annuity.}$$

Solution.

$(1.05)^5$	=	1.276281, compound amount of 1 at 5% for 5 periods.
$1.276281 - 1$	=	.276281, compound interest on 1 at 5% for 5 periods.
$.276281 \div .05$	=	5.52563, amount of annuity of 1 at 5% for 5 periods.
$\$200 \times 5.52563$	=	$1105.13, amount of annuity of $200 at 5% for 5 periods.

This is found to be a little more than the required amount; therefore the next trial rate should be less.

Second trial rate.—A trial rate of 4½% will be used.

$$200\left(\frac{(1.045)^5-1}{.045}\right) = \text{Amount of annuity.}$$

Solution.

$(1.045)^5$	=	1.2461819, compound amount of 1 at 4.5% for five years.
$1.2461819 - 1$	=	.2461819, compound interest on 1 at 4.5% for five years.
$.2461819 \div .045$	=	5.470709, amount of annuity of 1 at 4.5% for five years.
$\$200 \times 5.470709$	=	$1094.14, amount of annuity of 200 at 4.5% for five years.

This is found to be smaller than the amount in the problem, $1099.62.

INTERPOLATION OF AMOUNTS

Amount of annuity at 5%	$1105.13
Amount of annuity at 4½%	1094.14
Difference in amount caused by ½% difference in rates	$ 10.99
Amount of annuity at unknown rate	$1099.62
Amount of annuity at 4½%	1094.14
Difference in amount	$ 5.48

$\frac{548}{1099}$ of ½% = approximately ¼%.

4½% + ¼% = 4¾%, the required rate.

Problems.

1. Calculate the rates in the following:

	Amount of Annuity	*Rents*	*Periods*
(a)	$ 9.55	$ 1.	8
(b)	663.29	100.	6
(c)	5416.32	1000.	5
(d)	855.63	200.	4
(e)	9.21	1.	8
(f)	500.	50.	8

2. What rate of interest, convertible semiannually, for five years, will produce $750 as the amount of an annuity if the semiannual payments are $65 each?

3. If the present worth of a ten-year annuity is $811.09, and the annual payments are $100 each, what is the rate?

Chapter 29

SPECIAL ANNUITIES

Annuity due.—An "annuity due" is an annuity, the periodic payments of which are payable at the beginning of their respective periods. A comparison of the following tables will show the difference between an annuity due and an ordinary annuity, the payments of which are made at the end of each period.

TABLE

1	*2*	*3*	*4*
Year		*Amount of Ordinary Annuity Rents Due*	*Amount of Annuity Due Rents Due*
Jan. 1, 1918	Contract made	.....	1st Payment
Jan. 1, 1919		1st Payment	2nd Payment
Jan. 1, 1920		2nd Payment	3rd Payment
Jan. 1, 1921		3rd Payment	4th Payment
Jan. 1, 1922	Contract ends	4th Payment	.....

TABLE

1	*2*	*3*	*4*	*5*
End of Period	*4-Period Compound Amount*	*Amount of Ordinary Annuity*	*5-Period Compound Amount*	*Amount of Annuity Due*
1	1.00	1.00	1.00	1.06
2	1.06	2.06	1.06	2.1836
3	1.1236	3.1836	1.1236	3.374616
4	1.191016	4.374616	1.191016	4.637093
5	1.262477	5.637093	1.262477	
			5.637093	

Amount of an ordinary annuity for 5 periods..	5.637093
Deduct	1.000000
Amount of an annuity due for 4 periods.....	4.637093

From the illustration above it can be seen that the amount of an annuity due for 4 periods is 1. less than the amount of an ordinary annuity for 5 periods.

ANALYSIS OF THE AMOUNT OF AN ANNUITY DUE

Assumed Dates	*1923*	*1924*	*1925*	*1926*	*Total Value December 31, 1926*
January 1, 1923, Rent..	1.____	_____	_____	_____	1.262477
January 1, 1924, Rent..		1.____	_____	_____	1.191016
January 1, 1925, Rent..			1.____	_____	1.1236
January 1, 1926, Rent..				1.____	1.06
December 31, 1926, Annuity Due					4.637093

As shown above, the periodic rent draws interest from the beginning of the year in which it is deposited, and continues to draw interest until the due date. The sum of these periodic amounts is the amount of an annuity due.

As each payment of an annuity due is made at the beginning of a period, the final result is that the amount of an annuity due exceeds the amount of an ordinary annuity by the sum of the interest on the amount of the ordinary annuity for one period. The amount of the annuity due may be found in two ways:

First, by adding one period's interest to the amount of an ordinary annuity for the number of periods specified.

Second, by finding the amount of an ordinary annuity for one payment more than the number of payments specified and deducting one payment or rent from the total.

Procedure, first method.—(*a*) Compute the amount of an ordinary annuity at the given rate per cent and for the given rents.

(*b*) Multiply the amount of an ordinary annuity found in (*a*) by 1 plus the rate of interest per period expressed decimally.

Example.—What is the amount of an annuity, the payments of which are to be $100, made at the beginning of each year for four years, money being worth 6%?

Algebraic Formula.

$$\left[R\left(\frac{(1+i)^n-1}{i}\right)\right](1+i) = \text{Amount of annuity due.}$$

Arithmetical Substitution.

$$\left[100\left(\frac{(1.06)^4-1}{.06}\right)\right](1.06) = \text{Amount of annuity due.}$$

Solution.

$(1.06)^4 = 1.262477$, compound amount of 1 at 6% for 4 periods.
$1.262477 - 1 = .262477$, compound interest of 1 at 6% for 4 periods.
$.262477 \div .06 = 4.374616$, amount of an ordinary annuity of 1 at 6% for 4 periods.
$100 \times 4.374616 = 437.4616$, amount of an ordinary annuity of 100 at 6% for 4 periods.
$437.4616 \times 1.06 = \$463.71$, amount of an annuity due of 100.

Verification.

Assumed Dates			
Jan. 1, 1916	Contract made.		
	Rent	$100.00	$100.00
Jan. 1, 1917	Rent	100.00	
	Interest on $100 at 6%	6.00	106.00
	New principal		$206.00
Jan. 1, 1918	Rent	100.00	
	Interest on $206 at 6%	12.36	112.36
	New principal		$318.36
Jan. 1, 1919	Rent	100.00	
	Interest on $318.36	19.10	119.10
	New principal		$437.46
Jan. 1, 1920	Interest on $437.46	26.25	26.25
	Amount due		$463.71

Procedure, second method.—(*a*) Determine the amount of an ordinary annuity of 1 for the required number of periods plus 1.

(*b*) Deduct 1 from the result found in (*a*).

(*c*) Multiply the difference found in (*b*) by the number of dollars in each periodic rent to find the amount of the annuity due.

Algebraic Formula.

$$R\left(\frac{(1+i)^{n+1}-1}{i}-1\right) = \text{Amount of annuity due.}$$

Arithmetical Substitution.

$$100\left(\frac{(1.06)^{4+1}-1}{.06}-1\right) = \text{Amount of annuity due.}$$

Solution.

$(1.06)^5 = 1.338225$, compound amount of 1 for 5 periods at 6%.
$1.338225 - 1 = .338225$, compound interest of 1 for 5 periods at 6%.
$.338225 \div .06 = 5.63709$, amount of an ordinary annuity for 5 periods at 6%.
$5.63709 - 1 = 4.63709$, amount of an annuity due of 1 for 4 periods at 6%.
$\$100 \times 4.63709 = \$463.71$, amount of an annuity due for 4 years at 6%.

Problems.

Give formulas, solutions, and verifications for the following:

1. An annuity contract calls for a payment of $100 at the beginning of each year for five years, money being worth 4%, compounded annually. What is the value at the end of the fifth year?

2. A man agrees to deposit $300 in the bank the first of each quarter for five years, in favor of his son. If the bank allows 6% interest, compounded quarterly, what amount will the son receive five years hence?

3. A man agrees to deposit with a certain company $300 at the beginning of each six months, for the next ten years, and in return the company agrees to deed to him a house and lot. The rate of interest to be used is 4%, compounded semiannually. Calculate the cost of the house at the end of the ten-year period.

Present value of an annuity due.—The present value of an annuity due is the present or actual cash value at the date of the first payment of the payments to be produced by the annuity.

TABLE

	Value on Jan. 1, 1923	*1923*	*1924*	*1925*	*1926*
Jan. 1, 1923, Rent.....	1.	1.			
Jan. 1, 1924, Rent.....	.943396	_____	1.		
Jan. 1, 1925, Rent.....	.889996	_____	_____	1.	
Jan. 1, 1926, Rent.....	.839629	_____	_____	_____	1.
Total	3.673011				

Example.—What is the present value of an annuity due which has four annual payments of $1 each, at 6%?

TABLE OF COMPARISON OF PRESENT VALUES OF AN ORDINARY ANNUITY AND AN ANNUITY DUE

	Ordinary Annuity		*Annuity Due of*
No. of Rents	*of 4 Rents*	*of 3 Rents*	*4 Rents*
1	.943396	.943396	1
2	1.833392	1.833392	1.943396
3	2.673011	2.673011	2.833392
4	3.465105		3.673011

By multiplying a number in column 2 by 1 plus the rate per cent, (1.06), the product will be the number in column 4, corresponding to the same number of rents, thus: $3.46105 \times 1.06 = 3.673011$, the present value of an annuity due at 6% for 4 periods. Or, by the addition of one to the present value of an ordinary annuity, the sum will be the present value of an annuity due with an additional rent thus:

Present value of an ordinary annuity of 3 rents.....	2.673011
Adding 1	1.000000
Present value of annuity due for 4 periods.....	3.673011

To find the present worth of an annuity due.—Two methods may be used to find the present worth of an annuity due.

Example.—What is the present worth of an annuity of four payments of $100 each, if money is worth 6% per annum, and the payments are made at the beginning of each year?

Procedure, method 1.—(*a*) Compute the present value of an ordinary annuity, using the given number of dollars in the rents, the given rate per cent, and the given number of periods.

(*b*) Multiply the present value found in (a) by the rate per cent per period plus 1.

Algebraic Formula.

$$R\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)(1+i) = \text{Present worth of annuity due.}$$

Arithmetical Substitution.

$$100\left(\frac{1-\frac{1}{(1.06)^4}}{.06}\right)(1.06) = \text{Present worth of an annuity due.}$$

Solution.

$(1.06)^4$	=	1.262477,	compound amount of 1 at 6% for 4 periods.
$1 \div 1.262477$	=	.7920937,	present worth of 1 at 6% for 4 periods.
$1 - .7920937$	=	.2079063,	compound discount on 1 at 6% for 4 periods.
$.2079063 \div .06$	=	3.465105,	present worth of an ordinary annuity of 1.
100×3.465105	=	346.5105,	present worth of an ordinary annuity of 100.
$\$346.5105 \times 1.06$	=	$367.30,	present worth of an annuity due of $100.

Verification.

Assumed Dates			
Jan. 1, 1922	Present worth		$367.30
	Rent deducted	$100.00	100.00
	New principal		267.30
Jan. 1, 1923	Rent	100.00	
	Less interest on 267.30	16.04	
Balance to apply on principal		83.96	83.96
	New principal		183.34
Jan. 1, 1924	Rent	100.00	
	Less interest on 183.34	11.00	
Balance to apply on principal		89.00	89.00
	New principal		94.34
Jan. 1, 1925	Rent	100.00	
	Less interest on 94.34	5.66	
Balance to apply on principal		94.34	94.34
			$ .00

Procedure, method 2.—(*a*) Compute the present value of an ordinary annuity of 1 at the required rate and for one less than the required number of periods.

(*b*) To the present value of the annuity found in (*a*), add 1, the result of which is the present value of an annuity due for the required number of periods.

(*c*) Multiply the present value of an annuity due of 1 by the number of dollars in each rent.

Algebraic Formula.

$$R\left(\frac{1-\frac{1}{(1+i)^{n-1}}}{i}+1\right) = \text{Present worth of annuity due.}$$

Arithmetical Substitution.

$$100\left(\frac{1-\frac{1}{(1.06)^{4-1}}}{.06}+1\right)=\text{Present worth of an annuity due.}$$

Solution.

$(1.06)^{4-1}$ or $(1.06)^3$	=	1.191016, compound amount of 1 for 3 periods at 6%.
$1 \div 1.191016$	=	.839619, present worth of 1 for 3 periods at 6%.
$1 - .839619$	=	.160381, compound discount on 1 for 3 periods at 6%.
$.160381 \div .06$	=	2.6730, present worth of an ordinary annuity of 1 for 3 periods at 6%.
$2.6730 + 1$	=	3.6730, present worth of an annuity due of 1 for 4 periods at 6%.
$\$100 \times 3.6730$	=	$376.30, present worth of an annuity due of $100 for 4 years at 6%.

Problems.

Construct formulas, solutions, and verifications for Problems 1 and 2.

1. What is the present value of an annuity due in which $50 payments are to be made the first day of each six months for ten years, if money is worth 6%, interest compounded quarterly?

2. The rent of an annuity due is $500, payable semiannually, for ten years. If money is worth 4%, compounded semiannually, what is the value of the annuity at the date of the payment of the first rent?

3. The maximum sum insured under the soldiers' insurance with the Government is payable $57.50 per month for 20 years certain after death or disability. How can the equivalent sum payable in cash be found? Give the correct formula and explain why $12 \times 57.50 \times A_{20}$ differs from the equivalent cash sum. Given A_{20} at $3\frac{1}{2} = 14.21$.—*American Institute Examination.*

Note.—Are the payments made at the beginning or the end of each month? If payable each month, would the number of annuity periods be 20 or 240 (20×12)?

Rent of the amount of an annuity due.—The rent may be found by the following procedure:

Procedure (By method 2 of amount of annuity due).—(*a*) Compute the amount of an annuity due of 1 for the required number of periods and at the required rate per period.

(*b*) Divide the amount of an annuity due by the amount of an annuity due of 1, as computed in (*a*) to find the rent.

Example.—What sum, placed in the bank at the beginning of each year for four years, will amount to $100 at the end of four years, interest being compounded at 6%?

Algebraic Formula.

$$\frac{P}{\dfrac{(1+i)^{n+1}-1}{i}-1} = \text{Rent of the amount of an annuity due.}$$

Arithmetical Substitution.

$$\frac{100}{\dfrac{(1.06)^{4+1}-1}{.06}-1} = \text{Rent of the amount of an annuity due.}$$

Solution.

$(1.06)^5 = 1.338225$, compound amount of 1 for 5 periods at 6%.
$1.338225 - 1 = .338225$, compound interest on 1 for 5 periods at 6%.
$.338225 \div .06 = 5.6371$, amount of an ordinary annuity of 1 for 5 periods at 6%.
$5.6371 - 1 = 4.6371$, amount of an annuity due of 1 for 4 periods at 6%.
$\$100. \div 4.6371 = \$21.57$, rent of the amount of an annuity due for 4 years at 6%.

Verification.

Assumed Dates			
Jan. 1, 1922	Rent (amount at beginning of 1st year)...		$ 21.57
Jan. 1, 1923	Rent	$21.57	
	Interest due on $21.57 at 6%.....	1.29	22.86
	Amount at beginning of 2nd year.....		44.43
Jan. 1, 1924	Rent	21.57	
	Interest on $44.23 at 6%.....	2.67	24.24
	Amount at beginning of 3rd year.....		68.67
Jan. 1, 1925	Rent	21.57	
	Interest on $68.67 at 6%.....	4.12	25.69
	Amount at beginning of 4th year.....		94.36
	Interest on $94.36 at 6%.....		5.66
			100.02

Problems.

Construct formulas, solutions, and verifications for Problems 1 and 2.

1. What are the rents of an annuity due which amount to $10,000 in six years, if money is worth 4% per annum?

2. A man desires to know how much must be placed in the bank the first day of each six months' period for five years to accumulate $15,000, if the bank pays 4% interest, compounded semiannually.

3. *A* desires to know the payment, which is to be made on the first day of each quarter, for six years to accumulate enough to buy a home at that time worth $10,000, if the bank allows him 4% interest, compounded quarterly.

Rent of the present value of an annuity due.—The rent of the present value of an annuity due may be found by the following procedure:

Procedure (By method 2 of the present value of annuity due).—(*a*) Compute the present value of an annuity due of 1 for the required number of periods and at the given rate per cent.

(*b*) Divide the given present value of an annuity due by the present value of an annuity due of 1, as computed in (*a*).

Example.—A man owes $5000 and wishes to know the size of each of ten equal annual payments which will exactly cancel the debt, and pay the accrued interest due on the date of each payment. The first payment is to be made today. Money is worth 6% per annum.

Algebraic Formula.

$$\frac{P}{\left(1-\frac{1}{\frac{(1+i)^{n-1}}{i}}+1\right)} = \text{Rent of the present worth of an annuity due.}$$

Arithmetical Substitution.

$$\frac{5000}{\left(1-\frac{1}{\frac{(1.06)^{10-1}}{.06}}+1\right)} = \text{Rent of the present worth of an annuity due.}$$

Solution.

$(1.06)^9 = 1.689479$, compound amount of 1 for 9 periods at 6%.
$1. \div 1.689479 = .591898$, present worth of 1 for 9 periods at 6%.
$1 - .591898 = .408102$, compound discount on 1 for 9 periods at 6%.
$.408102 \div .06 = 6.8017$, present worth of an ordinary annuity for 9 periods at 6%.
$6.8017 + 1. = 7.8017$, present worth of an annuity due for 10 periods at 6%.
$\$5000 \div 7.8017 = \$640.89$, present worth of an annuity due of $5000.

Problem.

1. *A* purchases a farm for $10,000, with interest, and desires to pay for it in twelve equal annual payments, the first payment to be made immediately. What should be the number of dollars of each payment to cancel the debt and interest, money being worth 5%?

Effective interest on annuity due.—When interest is compounded more often than the payments are made, the effective rate should be found for a period to correspond with the periods of the annuity payments.

Procedure.—(*a*) Reduce the given interest rate to an effective rate for a period corresponding to the period of annuity payments.

(*b*) Using the effective rate found in (*a*) as the rate for each period, proceed by following the instructions on the preceding pages of this chapter.

Illustration of an amount of an annuity due in which the interest is compounded quarterly and the payments are made annually.

Example.—What is the amount of an annuity due, the annual payments being $100, for three years and the interest 6%, compounded quarterly?

Part 1.

Algebraic Formula. $\left(1 + \frac{j}{m}\right)^n$ = Effective rate.

Arithmetical Substitution. $\left(1 + \frac{.06}{4}\right)^4$ = Effective rate for an annuity period.

Solution to Part 1.

$.06 \div 4 = .015$, quarterly rate.
$1 + .015 = 1.015$, ratio of increase for 1 quarter.
$(1.015)^4 = 1.0613635$, compound amount of 1 for 4 periods.
$1.0613635 - 1. = .0613635$, effective interest for an annuity payment period.

Part 2.

Algebraic Formula.

$$R\left(\frac{(1+i)^{n+1}-1}{i}-1\right) = \text{Amount of annuity due.}$$

Arithmetical Substitution.

$$100\left(\frac{(1.0613635)^4-1}{.0613635}-1\right) = \text{Amount of annuity due.}$$

Solution to Part 2.

Substituting for i the rate found in part 1, the solution would be:

$(1.0613635)^4$	=	1.2689855,	compound amount of 1 for 4 periods at 6.13635%.
$1.2689855 - 1$	=	.2689855,	compound interest on 1 for 4 periods at 6.13635%.
$.2689855 \div .0613635$	=	4.3834,	amount of an ordinary annuity of 1 for 4 periods at 6.13635%.
$4.3834 - 1.$	=	3.3834,	amount of an annuity due for 3 periods at 6.13635%.
$\$100 \times 3.3834$	=	$338.34,	amount of annuity due of $100 for 3 years.

Problems.

1. What will be the amount at the end of three years of an annuity, the rents of which are $500, payable at the end of each year, the nominal rate being 5% per annum, compounded semiannually? Give formulas, solution and verification.

2. A company desires to know how much will be accumulated at the end of four years if $1000 is placed in a sinking fund at the end of each year. The fund bears a nominal rate of 8%, compounded semiannually. Give verification.

3. The J. Smith Corporation desires to accumulate a sinking fund of $10,000 to meet a bond issue of that amount at the end of 10 years. What will be the annual payments made on the last day of each year, if the fund is so placed that it will bear an annual rate of 8%, interest compounded semiannually? Construct columnar table showing payments, amount of fund, interest, and so forth.

4. The Davis Company has a bond issue of $20,000 coming due at the end of five years. The directors desire to know how much money must be placed in the bank at the end of each year in order that this fund may exactly cover the bond issue at the end of the required time. The bank pays 4% interest, compounded quarterly. Construct a columnar table.

5. What is the present worth of an ordinary annuity, the annual rents of which are $100, if the time is six years, money being worth 6%, compounded semiannually? Construct a columnar table.

6. What is the present worth of a contract in which a company agrees to pay $300 at the beginning of each year for five years, and is to be allowed interest at the rate of 8% per annum, interest compounded quarterly? Construct a columnar table.

Deferred annuity.—An annuity contract in which a number of periods are to expire before the periodic payments or rents are to begin, is a deferred annuity.

The amount of a deferred annuity would be the same as the amount of one which is not deferred, as the deferred periods will in no way change the value at the end of an annuity contract.

The present value of a deferred annuity is the value at the beginning of the deferred periods.

There are two methods of computing the present value of a deferred annuity. In each of these methods two distinct operations are necessary.

Procedure, method 1.—(*a*) Determine the present value of an ordinary annuity of 1 at the given rate and for the number of periods corresponding to the number of rents.

(*b*) Multiply the present value of 1 found in (*a*) by the number of dollars in each rent.

(*c*) Multiply the present worth of the annuity found in (*b*) by the present value of 1 for the number of deferred periods.

Example.—Find the present value of an annuity contract which calls for four equal annual payments of $100 each, the first rent to be paid at the end of the seventh year. Interest is calculated at 6%.

Analyzed: (*a*) Ordinary annuity for four years.
(*b*) Deferred for six years.

Algebraic Formula.

Let: n equal the number of rents.
m equal the number of deferred periods.

$$R\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right]\left[\frac{1}{(1+i)^m}\right] = \text{Present worth of deferred annuity.}$$

Arithmetical Substitution.

$$100\left[\frac{1-\frac{1}{(1.06)^4}}{.06}\right]\left[\frac{1}{(1.06)^6}\right] = \text{Present worth of deferred annuity.}$$

Short Solution, Part 1.

1. ÷ 1.262477 = .7920937, present worth of 1 due at the end of 4 years at 6%.
1. — .7920937 = .2079063, compound discount on 1 due at the end of 4 years at 6%.
.2079063 ÷ .06 = 3.465105, present worth of annuity of 1 for 4 years at 6%.
$100. × 3.465105 = $346.51, present worth of annuity of $100 for 4 years.

Short Solution, Part 2.

$(1.06)^6$ = 1.418519
1. ÷ 1.418519 = .7049605, present worth of 1 due at the end of 6 years at 6%.
$346.51 × .704605 = $244.28, present worth of deferred annuity.

Verification, Part 1.

Assumed Dates			
Jan. 1, 1923	Present worth of deferred annuity..		$244.28
	Multiply by $(1.06)^6$.....		1.418519
Jan. 1, 1929	Value at end of deferred period....		346.51
Dec. 31, 1929	Rent	$100.00	
	Interest on 346.51 at 6%.....	20.79	
	Amortization		79.21
			267.30
Dec. 31, 1930	Rent	100.00	
	Interest on 267.30 at 6%.....	16.04	83.96
			183.34
Dec. 31, 1931	Rent	100.00	
	Interest on 183.34 at 6%.....	11.00	89.00
			94.34
Dec. 31, 1932	Rent	100.00	
	Interest on 94.34 at 6%.....	5.66	94.34

Procedure, method 2.—(*a*) Compute the present value of an ordinary annuity of 1 at the given rate per cent, for a number of periods equal to the sum of the number of periods of the annuity plus the number of deferred periods.

(*b*) Compute the present value of an ordinary annuity of 1, at the given rate and for the number of periods equal to the deferred periods.

(*c*) Find the difference between the present values found in (*a*) and (*b*).

(*d*) Multiply the difference in the present values of annuities found in (*c*) by a number equal to the number of dollars in each rent.

Algebraic Formula.

$$R\left[\left(\frac{1-\frac{1}{(1+i)^{m+n}}}{i}\right)-\left(\frac{1-\frac{1}{(1+i)^{n}}}{i}\right)\right] = \text{Present value of deferred annuity.}$$

Arithmetical Substitution.

$$100\left[\left(\frac{1-\frac{1}{(1.06)^{10}}}{.06}\right)-\left(\frac{1-\frac{1}{(1.06)^{6}}}{.06}\right)\right] = \text{Present value of deferred annuity.}$$

Solution.

$(1.06)^{10}$	=	1.790847,	compound amount of 1 at 6% for 10 periods.
$1 \div 1.79084$	=	.5583948,	present worth of 1 at 6% for 10 periods.
$1 - .5583948$	=	.4416052,	compound discount on 1 at 6% for 10 periods.
$.4416052 \div .06$	=	7.360087,	present value of annuity of 1 at 6% for 10 periods.
$(1.06)^{6}$	=	1.418519,	compound amount of 1 at 6% for 6 periods.
$1 \div 1.418519$	=	.7049605,	present worth of 1 at 6% for 6 periods.
$1 - .7049605$	=	.2950395,	compound discount on 1 at 6% for 6 periods.
$.2950395 \div .06$	=	4.917324,	present value of annuity of 1 at 6% for 6 periods.
$7.360087 - 4.917324$	=	2.442763,	difference in present value of annuities of 1 for 10 periods and 6 periods.
$\$100 \times 2.442763$	=	$\$244.28$,	present value of deferred annuity of \$100.

Problems.

Find the present worth of the following deferred annuities:

	Payments	Rate	Number of Rents	Years Deferred
1.	$100 Annually	6%	4	4
2.	400 "	4%	8	6
3.	250 Semiannually	6%	6	4

Note.—If interest is compounded semiannually, the same condition usually prevails during the deferred period.

4.	$ 300 Semiannually	8%	8	8
5.	200 "	6%	8	10
6.	1250 Quarterly	8%	12	20

7. What is the present value of an insurance policy in which the company agrees to pay to the insured a monthly installment of $25 for ten years, the first payment to be made five years hence? Money is worth 6%, convertible monthly during the annuity period, and annually during the deferred period.

8. A company is issuing $100,000, 4%, 20-year bonds, which it wishes to pay at maturity by means of a sinking fund in which equal annual deposits are to be made. The board of directors wishes to assume that this fund will earn 5½% interest for the first five years, 5% for the next five years, and 4% for the last ten years. What is the annual deposit required?

Given:	5½%	5%	4%
S_5	5.581	5.526	5.416
S_{10}	12.875	12.578	12.006
$(1+i)^5$	1.307	1.276	1.217
$(1+i)^{10}$	1.708	1.629	1.480

—*American Institute Examination.*

9. A lease has five years to run at $1000 a year, payable at the end of each year, with an extension for a further five years at $1200 a year. On a 6% basis, what sum should be paid now in lieu of the ten years' rent? (V^5 at 6% = .7472582.)—*American Institute Examination.*

Perpetuity.—A perpetuity is defined as a series of periodic payments which are to run indefinitely. This form of annual payment is most often found as the effect of the

establishment of an endowment fund, the rents of which are to be used for a special purpose. As the endowment found is never to be returned, the amount will have no meaning, but its present worth will be the value of the periodic payment divided by the prevailing rate of interest.

Procedure.—(*a*) Divide the periodic rent by the rate per cent expressed decimally, to find the present value of the perpetuity.

Example.—How much money must be placed in a fund in order that the annual income therefrom at 6% will amount to $300?

Algebraic Formula.	*Arithmetical Substitution.*
$\frac{\text{Rent}}{i} = \text{Perpetuity.}$	$\frac{\$300}{.06} = \$5000.$

Solution.

$300 ÷ .06 = $5000, endowment fund.

Replacement of assets.—The question arises as to the present value of a fund necessary to pay the first cost of an asset and to provide an amount necessary to renew the asset indefinitely. The occasion for this calculation occurs most frequently when an endowment fund provides for the periodic replacement of a useful memorial, such as a building, or bridge.

Procedure.—(*a*) Divide the cost of the asset by the rate per cent expressed decimally to find the capitalized cost.

(*b*) Divide the capitalized cost of the asset found in (*a*) by the amount of an ordinary annuity of 1 at the given rate per cent and for the time, expressed in periods, which represents the life of the asset.

(This will give the amount of an endowment fund necessary to replace the asset continually at the end of each five years.)

(*c*) Add the cost of the asset to the amount of the endowment fund found in (*b*). This will provide for the first cost, as well as the replacement fund.

Example.—What is the size of a fund which is necessary to erect a memorial costing $2000, and to provide for replacement at the end of each five years, money being worth 6%?

Algebraic Formula.

$$\frac{\frac{\text{Cost}}{i}}{\frac{(1+i)^n-1}{i}} + \text{Cost} = \text{Size of fund.}$$

Arithmetical Substitution.

$$\frac{\frac{2000}{.06}}{\frac{(1.06)^5-1}{.06}} + 2000 = \text{Size of fund.}$$

Solution.

$2000 \div .06$	$= 33{,}333.33,$	capitalized cost of asset.
(1.06)	$= 1.3382256,$	compound amount of 1 at 6% for 5 years.
$1.3382256 - 1$	$= .3382256,$	compound interest on 1 at 6% for 5 years.
$.3382256 \div .06$	$= 5.637093,$	amount of ordinary annuity of 1 at 6% for 5 years.
$33333.33 \div 5.637093$	$= 5{,}913.21,$	sum available for investment at compound interest.
$\$5913.21 + \$2000$	$= \$7{,}913.21,$	total fund.

Verification.

$2000, present cost of memorial.

$\$7913.21 - \$2000 = \$5913.21$, sum available for investment at compound interest.

$\$5913.21 \times 1.3382256 = \$7913.21$, total fund five years hence.

Problems.

1. What is the capitalized cost of a memorial building, the cost of which is $150,000? It is estimated the building will last 40 years, and money is worth 5%.

2. A philanthropist desires to know how much of a fund is necessary to erect a college building and provide for indefinite renewals, if the cost is $100,000, and the estimated life is 35 years, money being worth 4%. Assume that repairs and renewals are to be made from some other fund.

Perpetuity for renewals and replacements of assets.—Many endowments provide for annual and ordinary expenses, as well as replacement of assets.

Procedure.—(*a*) Divide the cost of the asset by the rate per cent expressed decimally to find the capitalized cost.

(*b*) Divide the capitalized cost of the asset by the amount of an ordinary annuity of 1 at the given rate per cent for as many periods, expressed in years as the life of the asset, in order to find the replacement fund.

(*c*) Divide the annual upkeep by the rate per cent to find the capitalized value of the upkeep fund.

(*d*) Add the building fund, the cost of building, and the upkeep fund to find the total funds necessary.

Example.—The Rutledge Home cost $75,000. The annual expenses of running the institution, including repairs and upkeep, are estimated at $5000. If the life of the building is 50 years and money is worth 4%, what sum is necessary to provide for same indefinitely?

Algebraic Formula.

$$\frac{\frac{\text{Cost}}{i}}{\frac{(1+i)^n-1}{i}} + \text{Cost} + \frac{\text{Upkeep}}{i} = \text{Sum.}$$

Arithmetical Substitution.

$$\frac{\frac{75{,}000}{.04}}{\frac{(1.04)^{50}-1}{.04}} + 75{,}000 + \frac{5{,}000}{.04} = \text{Sum.}$$

Solution, Part 1.

$75,000 \div .04 = 1,875,000$
$\$1,875,000 \div \$152.66708 = \$12,281.63$, building replacement fund.

Solution, Part 2.

$\$75,000 =$ Cost of building.

Solution, Part 3.

$\$5000 \div .04 = \$125,000$, upkeep fund.

Summary.

$ 12,281.63	Building fund.
75,000.00	Cost of building.
125,000.00	Upkeep fund.
$212,281.63	Total funds necessary.

Problems.

1. What is the capitalized value of a school building, the original cost of which is $200,000, and the estimated life, 40 years? The annual upkeep is $15,000, and money is worth 5%.

2. What sum will be required not only to build and replace a bridge, but to provide for the upkeep thereof indefinitely? The cost is $25,000, and by a conservative estimate will last 25 years; the upkeep including the watchman's salary will be $1,800. Money is worth 5½% annually.

3. An "Old People's Home" is to be built and maintained by a philanthropist. It is estimated that the building will cost $100,000, and that the life of the building will be 75 years. The estimated annual income and costs will be as follows:

Donations	$4000
Miscellaneous Service Charges	1000
Heat	2000
Matron	1800
Help	3000
Food	5000
Medical Attention	1200
Miscellaneous	1000

Find the value of the endowment at 4% interest, convertible annually.

Chapter 30

BOND AND BOND INTEREST VALUATION

Definitions.—A bond is a promise to pay a specified sum of money at a determinable future date. It differs from a note, in general, in that it is usually a long term obligation. Bonds are usually issued by a city, state, nation, or corporation, and seldom by individuals. The sum written in the body of the instrument is known as the *face, par,* or *nominal par.* This sum is the amount on which the interest is calculated. Bonds usually have a par value of $100, $500, or $1000.

It is not necessary for bonds to bear interest, but they usually do. Interest payments may be made annually, semi-annually or quarterly. These interest payments are known as *nominal interest,* or *cash interest.*

Effective or *investment interest* rates are terms used to designate the rate per cent earned on the actual money invested. The cash rate and effective rate will not be the same unless the bond is purchased at par. Quotations are sometimes made "on a basis," which means at an effective rate on the money invested. The price of a bond will be either above or below par according as the cash rate is above or below the effective rate.

A bond is said to be *redeemed* when it is bought back by the company issuing it.

If the market price of a bond is more than the par value, it is said to be *above par,* or if less, the bond is said to be *below par.* Bonds sold above par are said to be sold "at a premium," and if sold below par are sold "at a discount."

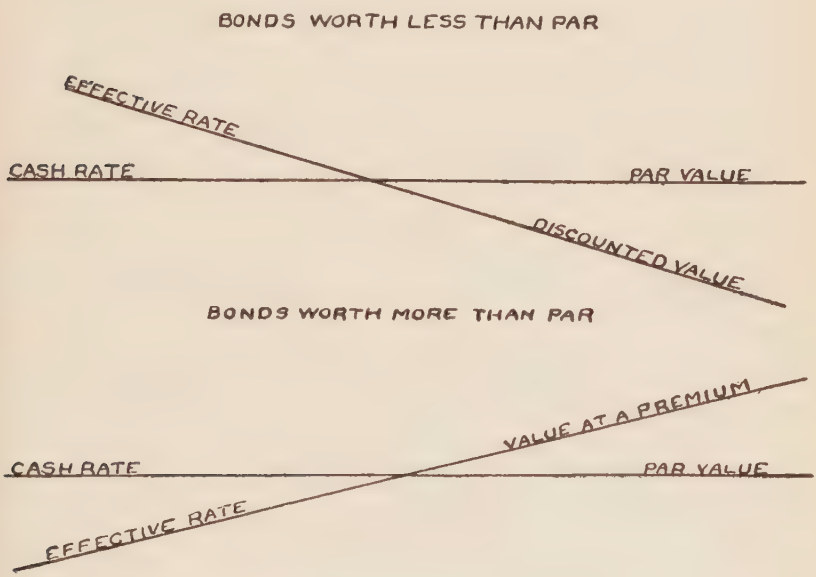
Bonds are usually sold on the open market for what they will bring. Whether they bring more or less than par depends on many things:

1. The cash or coupon rate of interest.
2. The redemption price.
3. The current rate of interest.
4. The length of time until maturity.
5. The character of the security.

In the calculations of the price of bonds, only the first four of the above reasons can be mathematically considered.

Bonds sold at par.—It is apparent that if a man desires to buy a bond of a face value of $1000, and this bond bears a cash rate of interest exactly the same as that which his money is worth on the market for other securities of the same general class, he will be willing to pay $1000 for it. In such a case, the interest he will receive from this bond will be equivalent to the interest he would receive from any other investment of the same general class. But if he considered the purchase of another bond of $1000, which had a cash rate of interest lower than that of other investments of the same general class, it is apparent that he would not pay a full thousand dollars for the bond. And again, if securities of the same general class paid a higher rate per thousand, he would be willing to pay more than the par value for the bond.

This may be illustrated graphically:



If a bond is purchased at a discount, the effective rate is higher than the cash rate for two reasons: first, the investment is less than par, and second, the bond holder makes a profit equal to the difference between the cost and the par to be collected at maturity. On the other hand, if a bond is pur-

chased at a premium the effective rate is lower than the cash rate for two reasons: first, the investment is more than par, and second the investor loses the difference between the price paid and par to be collected at maturity.

In business, usually, one of two questions is asked about bonds, "What price should be paid for bonds so as to produce a certain investment rate of interest?" or "What rate will a bond, purchased at a certain price, produce?"

Use of bond tables.—Bond tables are generally used to calculate the cost. An illustration of a bond table is given below to familiarize the student with its use.

Example.—What is the purchase price of a 5-year, 5% bond (interest payable semiannually), bought on a 6% basis?

Referring to the table, the answer is found to be $95.73.

TABLE OF BOND VALUES

FIVE YEARS, INTEREST PAYABLE SEMIANNUALLY

Effective Rate	NOMINAL RATES						
	3	*3½*	*4*	*4½*	*5*	*6*	*7*
5½.....	89.20	91.36	93.52	95.68	97.84	102.16	106.48
5⅝.....	88.70	90.85	93.00	95.16	97.31	101.61	105.92
5¾.....	88.20	90.34	92.49	94.63	96.78	101.07	105.37
5⅞.....	87.70	89.84	91.98	94.12	96.26	100.53	104.81
6	87.20	89.34	91.47	93.60	95.73	100.00	104.27

Problems.

1. What is the purchase price, or value of a 5%, $1000, 5-year bond, interest payable semiannually, bought on a 5½% basis?

2. State the value of a 6% bond for $100 for 5 years, interest payable semiannually, purchased on a basis of 5¾%.

3. What price can be paid for city bonds of $1000 each, due in 5 years, interest at 7%, payable semiannually, bought on a 6% basis?

If no bond tables are available, the values of bonds purchased at a discount or at a premium may be found by either of two methods of calculation.

Bonds sold at a discount.—Discount on bonds is the difference between the cost (which is below par) and the par value.

Two scientific methods of calculation are given below:

Method 1, procedure.—(*a*) Compute the present value of the par of the bond at the investment rate of interest and for the number of periods between the date of purchase and the maturity of the bond.

(*b*) Compute at the effective interest rate the present value of an annuity for a number of periods equal to the number of bond-interest or cash-interest payments, the rents of the annuity to be the same size as the periodic cash or bond interest payments.

(*c*) Add the present value of the par of the bond found in (*a*) and the present value of the annuity found in (*b*). The sum will be the present value of the bond.

Example.—What will be the cost of a 5% bond for $100, payable in four years, bought so that it will produce 6% effective interest, payable semiannually?

Algebraic Formula.

$$\left(\begin{matrix}\text{Par}\\ \text{value}\end{matrix}\right)\left(\frac{1}{(1+i)^n}\right)+\left(\begin{matrix}\text{Each}\\ \text{cash interest}\end{matrix}\right)\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)=\text{Cost.}$$

Arithmetical Substitution.

$$100\left(\frac{1}{(1.03)^8}\right)+2.50\left(\frac{1-\frac{1}{(1.03)^8}}{.03}\right)=\text{Cost.}$$

Solution.

Part 1.

Finding the present value of the par of the bond:

$(1.03)^8$	= 1.2667701,	compound amount of 1 at 3% for 8 periods.
$1. \div 1.2667701$	= .7894092,	present value of 1 at 3% for 8 periods.
$\$100 \times .7894092$	= $78.94,	present worth of $100.00 at 3% for 8 periods.

Part 2.

Finding the present value of an annuity of the $2.50 interest payments:

.7894092	=	Present value of 1 as found above.
$1. - .7894092$	= .2105908,	compound discount on 1 at 3% for 8 periods.
$.2105908 \div .03$	= 7.01969,	present value of an annuity of 1.
$\$2.50 \times 7.01969$	= $17.549,	present value of an annuity of $2.50.

Part 3.

Adding:

$78.94 + $17.55 = $96.49, cost of bond.

A convenient and condensed statement showing for each period, the carrying value, the accumulation of discount, coupon interest, and the effective interest may be made thus:

TABLE

End of Period	*Effective Interest*	*Coupon Interest*	*Accumulation of Discounts*	*Carrying Value*
				96.49
1.....	2.89	2.50	.39	96.88
2.....	2.91	2.50	.41	97.29
3.....	2.92	2.50	.42	97.71
4.....	2.93	2.50	.43	98.14
5.....	2.94	2.50	.44	98.58
6.....	2.96	2.50	.46	99.04
7.....	2.97	2.50	.47	99.51
8.....	2.99	2.50	.49	100.00

Method 2.—The theory of this method is that the $2.50 interest received will offset $2.50 of the $3.00 expected; therefore, the cost of the bond will be the par value of the bond less the present value at the effective interest rate of an annuity of $.50.

Procedure.—(*a*) Compute the present value of an annuity of 1 at the effective interest rate for a number of periods equal to the number of periods that the bond has yet to run.

(*b*) Calculate the difference between the number of dollars of income per period at the desired rate and at the cash rate, based in both cases on the par of the bonds.

(*c*) Multiply the present value of the annuity found in (*a*) by the difference found in (*b*); the result will be the discount on the par value of a bond.

(*d*) Deduct the discount found in (*c*) from the par value of the bond, and the result will be the price.

Algebraic Formula.

$$\left(\begin{matrix}\text{Par}\\ \text{value}\end{matrix}\right) - \left[(E-C)\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)\right] = \text{Cost of bond.}$$

Arithmetical Substitution.

$$100 - \left[\left(3.00 - 2.50 \right) \left(\frac{1 - \frac{1}{(1.03)^8}}{.03} \right) \right] = \text{Cost of bond.}$$

Solution.

Part 1.

$(1.03)^8 = 1.2667701$, compound amount of 1 at 3% for 8 periods.
$1. \div 1.2667701 = .7894092$, present value of 1 at 3% for 8 periods.
$1. - .7894092 = .2105908$, compound discount on 1 at 3% for 8 periods.
$.2105908 \div .03 = 7.01969$, present value of an annuity of 1.

Part 2.

$3.00 - 2.50 = .50$, difference between effective and cash interest for 1 period.
$7.01969 \times .50 = \$3.51$, discount on bond.

Part 3.

$\$100. - \$3.51 = \$96.49$, cost of bond.

Verification.

First Period:		
Cost of bond		$ 96.49
Interest at 3% on $96.49	$2.89	
Coupon interest	2.50	
Accumulation of discount	.39	.39
		96.88
Second Period:		
Interest at 3% on $96.88	2.91	
Coupon interest	2.50	
Accumulation of discount		.41
		97.29
Third Period:		
Interest at 3% on $97.29	2.92	
Coupon interest	2.50	
Accumulation of discount		.42
		97.71
Fourth Period:		
Interest at 3% on $97.71	3.43	
Coupon interest	2.50	
Accumulation of discount		.43
		98.14
Fifth Period:		
Interest at 3% on $98.14	2.94	
Coupon interest	2.50	
Accumulation of discount		.44
		98.58

Sixth Period:		
Interest at 3% on $98.58	2.96	
Coupon interest	2.50	
Accumulation of discount		.46
		99.04
Seventh Period:		
Interest at 3% on $99.04	2.97	
Coupon interest	2.50	
Accumulation of discount		.47
		99.51
Eighth Period:		
Interest at 3% on $99.51	2.99	
Coupon interest	2.50	
Accumulation of discount		.49
Par of bond		$100.00

Problems.

1. What should be the purchase price of a $1000, 5-year, 6% bond, bought so that it will produce 7% interest on the investment? (Interest payable semiannually.)

2. If money is worth 6%, semiannual interest, what should be the purchase price of five $100 bonds, bearing a cash rate of 4%, and having 20 years to run?

3. Construct a table in columnar form showing the carrying value, cash interest, the effective interest, and amortization of a 6-year, 6% bond of $500, bought on a 7% basis (interest payable semiannually).

4. Construct a columnar table for a bond which has a face value of $500, interest payable quarterly at 6%, bought on a basis to produce 8%, convertible quarterly, for 3 years.

5. Show in columnar form the carrying value, the cash interest, the effective interest, and the accumulation of discount for a 4-year bond, the par value of which is $1000. The bonds bear 5% interest, payable semiannually; the effective rate is 6%, convertible semiannually.

Bonds sold at a premium.—As stated before, if the effective rate is less than the coupon rate, the bond will sell at a premium, or will be priced above par. When bonds sell at a premium, part of the money received for each coupon will be used to cancel part of the premium paid for the bond.

As in "Discount on Bonds," two methods of calculation are in common use. Both are explained below.

Method 1, procedure.—The procedure under method 1 of finding the present value of a bond when the purchase price is at a premium is the same as the procedure under method 1 of finding the price of bonds purchased at a discount.

Example.—What price may be paid for a \$100, 6%, 4-year bond, in order that the investor may realize 5% on his investment? Interest coupons are payable semiannually.

Algebraic Formula.

$$\left(\begin{matrix}\text{Par}\\ \text{value}\end{matrix}\right)\left(\frac{1}{(1+i)^n}\right) + \text{Cash interest}\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right) = \text{Value of bond.}$$

Arithmetical Substitution.

$$100\left(\frac{1}{(1.025)^8}\right) + 3\left(\frac{1-\frac{1}{(1.025)^8}}{.025}\right) = \text{Value of bond.}$$

Solution.

Part 1.

Finding the present worth of the face of the bond:

$(1.025)^8 = 1.2184029$, compound amount of 1 at 2½% for 8 periods.
$1. \div 1.2184027 = .8207466$, present worth of 1 at 2½% for 8 periods.
$\$100 \times .8207466 = \$82.07$, present worth of face of bond.

Part 2.

Finding the present worth of the annuity of \$3:

$1. - .8207466 = .1792534$, compound discount on 1 at 2½% for 8 periods.
$.1792534 \div .025 = 7.1701372$, present worth of an annuity of 1 at 2½% for 8 periods.
$\$3. \times 7.1701372 = \$21.51$, present worth of an annuity of \$3.

Part 3.

Adding:

$\$82.07 + \$21.51 = \$103.58$, cost of bond.

COLUMNAR TABLE SHOWING VERIFICATION

End of Period	*Effective Interest*	*Coupon Interest*	*Amortization*	*Carrying Value*
				103.58
1.....	2.59	3.00	.41	103.17
2.....	2.58	3.00	.42	102.75
3.....	2.57	3.00	.43	102.32
4.....	2.56	3.00	.44	101.88
5.....	2.55	3.00	.45	101.43
6.....	2.54	3.00	.46	100.97
7.....	2.52	3.00	.48	100.49
8.....	2.51	3.00	.49	100.00

Method 2, procedure.—(*a*) Compute the present value of an annuity of 1 at the investment rate of interest and for the number of periods equal to the unexpired number of periods of the bond.

(*b*) From the number of dollars of one cash interest subtract the number of dollars found by multiplying the par of the bond by the effective rate of interest.

(*c*) Multiply the present value of the annuity found in (*a*) by the number of dollars of the excess of the investment interest found in (*b*). The result is the premium on a bond.

(*d*) Add the par value of the bond and the premium found in (*c*) to obtain the present value of a bond at a premium.

Algebraic Formula.

$$\left[(C-E)\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)\right] + (\text{Par}) = \text{Value of bond.}$$

Arithmetical Substitution.

$$\left[(3.00-2.50)\left(\frac{1-\frac{1}{(1.025)^8}}{.025}\right)\right] + 100 = \text{Value of bond.}$$

Solution.

Part 1.

1.025	=	1.2184029, compound amount of 1 at 2½% for 8 periods.
1. ÷ 1.2184029	=	.8207466, present worth of 1 at 2½% for 8 periods.
1. — .8207466	=	.1792534, compound discount on 1 at 2½% for 8 periods.
.1792534 ÷ .025	=	7.1701372, present value of an annuity of 1 at 2½% for 8 periods.
$.50 × 7.1701372	=	3.58, present value of an annuity of $.50.

Part 2.

Adding:

$100. + $3.58 = $103.58.

Verification.

First Period:		
Cost of bond		$103.58
Coupon interest	$3.00	
Less 2½% interest on $103.59	2.59	
Amortization of premium		.41
		103.17
Second Period:		
Coupon interest	3.00	
Less 2½% interest on $103.18	2.58	
Amortization of premium		.42
		102.75
Third Period:		
Coupon interest	3.00	
Less 2½% interest on $102.76	2.57	
Amortization of premium		.43
		102.32
Fourth Period:		
Coupon interest	3.00	
Less 2½% interest on $102.33	2.56	
Amortization of premium		.44
		101.88
Fifth Period:		
Coupon interest	3.00	
Less 2½% interest on $101.89	2.55	
Amortization of premium		.45
		101.43
Sixth Period:		
Coupon interest	3.00	
Less 2½% interest on $101.44	2.54	
Amortization of premium		.46
		100.97
Seventh Period:		
Coupon interest	3.00	
Less 2½% interest on $100.98	2.52	
Amortization of premium		.48
		100.49
Eighth Period:		
Coupon interest	3.00	
Less 2½% interest on $100.49	2.51	
Amortization of premium		.49
Par of bond		$100.00

Problems.

1. What should be paid for a 10-year, 5% bond, interest payable semiannually, par value of bond $1000, bought on a 4% basis?

2. What should be the price paid for ten $1000 Minneapolis 6% semiannual interest bonds, maturing in 10 years, bought so as to yield 4%?

3. Construct a columnar table showing cash interest, effective interest, carrying value, and amortization of three $500, 8% bonds, which have four years yet to run; interest payable semiannually, and bought on a 6% basis.

4. A $10,000, 5%, semiannual coupon bond is bought on a 4% basis, due 1½ years hence. What did it cost?—*American Institute Examination.*

5. Fred Humiston left a will in which he directed that the income from his estate should be paid to his widow, and that the principal should be held in trust for a grandson. He also stipulated that the corpus should not be impaired by the purchase of bonds by the trustee at a premium, and directed that such premiums paid by the trustee should be amortized.

Humiston died on April 1, 1920. The estate contained five $1000 bonds of the *X Y Z* Company, bearing 6% interest, payable July 1st and January 1st. The bonds were due on July 1, 1925, and were inventoried at 104½.

On July 1, 1920, the trustee purchased five more of the same bonds on a 5% basis.

Compute the price paid by the trustee for the bonds. The present value of $1, due ten periods hence at 2½%, is $.781198402.

Prepare a statement showing the income payable to the widow on July 1, 1920, January 1, 1921, and July 1, 1921. —*American Institute Examination.*

6. What is the difference in the purchase price of two $1000, 20-year bonds, bought to yield 6%? The one bond has a semiannual coupon of $25, while the other has a semiannual coupon of $35.

7. Find the cost of the following bond issue:

$54,500

DAKOTA COUNTY, MINN.

4½%

TRUNK HIGHWAY REIMBURSEMENT

BONDS

Dated July 1, 1922. Due July 1, 1932.

FINANCIAL STATEMENT

Assessed Valuation (estimated)	$110,000,000.00
Assessed Valuation, 1921	27,116,112.00
Total Bonded Debt	515,551.07

Population, 1920 Census, 28,967.

Exempt from Federal Income Tax.

Tax Exempt in Minnesota.

Price to Net 4.30%.

Bond values between interest dates.—In finding the value of bonds heretofore, even periods were used in the calculation of interest. These calculations will not show the correct values at any other time than at interest dates, for two reasons: first, there will be accrued interest at the cash rate, and, second, the amortization varies between parts of interest period.

Interest accrued.—The accrued interest must always be considered in finding the accurate value of investments. Bond discount or premium may be treated in two ways:

1. The practical method is to calculate the amount of discount or premium for the even periods by the scientific method of amortization, and the amount applicable between periods by interpolation.

2. Theoretically, the rate of increase is effective at any moment during a period in the same ratio as it is during any number of periods; that is, the accumulation of discount or amortization of premium gradually increases or decreases proportionately as time increases or decreases.

Amortization between interest dates.—For practical purposes, the amortization between interest dates may be calculated on a proportional basis, by means of interpolation. This gives a fair degree of accuracy. By the use of bond tables, the amount may be readily found.

Illustration of the practical process of the calculation of the value of a bond bought at a discount.—

Procedure.—(*a*) By use of bond tables, or formulas previously given, determine the value of the bond at the interest date just preceding the purchase date, and the value of the bond at the next interest date subsequent to the purchase date.

(*b*) Calculate the discount for one period by finding the difference in the value determined in (*a*).

(*c*) Calculate such a proportional part of the discount of one period found in (*c*) as the fractional part of the period which has elapsed is to the total period.

(*d*) Add the discount of the fractional period, found in (*c*), to the value of the bond at the interest date just preceding the purchase date.

(*e*) Determine the accrued interest on the par of the bond for the time expired since the last interest date at the rate specified in the bond.

(*f*) Add the accrued interest found in (e), to the sum found in (*d*); the result will be the value of the bond and interest.

Example.—On March 1, 1923, what should be the value of a $1000, 4½% bond, due January 1, 1928? Interest coupons are payable January 1, and July 1, money being worth 6%, compounded semiannually.

Solution.

Value 9 periods before maturity at 6%	$941.60
Value 10 periods before maturity at 6%	936.02
Accumulation of discount during 1 period	$ 5.58

INTERPOLATION

Accumulation of discount during 1 period	$ 5.58
As two months of the six months period has elapsed, the simple proportional part of the discount accumulated will be ⅓ of $5.58 or $1.86.	
Value 10 periods before maturity	$936.02
Add 2 months accumulation of discount	1.86
Value 9 periods and 4 months before maturity	937.88
Add accrued portion of next interest coupons	7.50
Total value of bond and interest	$945.38

Theoretical accumulation of discount illustrated.—As the discount accumulates at the rate of 3% for one whole period, and as exactly one third of the period of time has expired, the accumulation may be expressed by a fraction:

$$\frac{(1.03)^{\frac{1}{3}} - 1.}{(1.03) - 1.} \times 5.58.$$

Extracting 3rd root of (1.03), $\frac{(1.009902) - 1.}{(1.03) - 1.} \times 5.58.$

Simplifying, $\frac{.009902}{.030000} \times 5.58 = 1.842.$

Accumulation of discount by *interpolation* (as above)	= 1.86
Accumulation of discount by *theoretical process*	= 1.842
Error by interpolation in the practical process of calculation	= .018

Illustration of the practical process of the calculation of the value of a bond bought at a premium.—

Procedure.—(*a*) Determine from bond tables the value of the bond at the interest date just preceding the purchase date, and the value of the bond at the first interest date subsequent to the purchase date.

(*b*) Determine the premium for one period by finding the difference in the values in (*a*).

(*c*) Calculate such a proportional part of the premium for one period as the expired part of the period is of a whole period.

(*d*) Deduct the premium found in (*c*) from the value of the bond at the interest date just preceding the purchase date.

(*e*) Determine the accrued interest on the par of the bond for the expired fractional part of the period.

(*f*) Add the accrued interest found in (*e*) to the answer found in (*d*); the sum will be the purchase price of the bond and accrued interest.

Example.—On May 1, 1923, what should be paid for a $1000, 6% bond, due January 1, 1927? Interest coupons are payable January and July, and money is worth 5% semi-annually.

Solution.

Value 8 periods before maturity at 5%	$1,035.85
Value 7 periods before maturity at 5%	1,031.75
Amortization of premium during one period	4.10

INTERPOLATION

Amortization of premium during one period	$4.10	
As 4 months of the 8th period have expired, the proportional part of the 6 months period is two-thirds. Therefore, the amortization which has taken place is two-thirds of $4.10 or $2.73.		
Value 8 periods before maturity		$1,035.85
Deduct 4 months amortization of premium		2.73
Value 7 periods and 2 months before maturity		1,033.12
Interest at 6% for one period	$30.00	
As four months have expired since any interest has been paid, there would be due four-sixths or two-thirds of $30		20.00
Value of bond and accrued interest		$1,053.12

Problems.

1. What price should be paid for a $100 bond, plus accrued interest? The bond bears 6% interest, payable semiannually, and is due in 4 years and 4 months. It should be bought at a price to produce 5% interest on the investment.

2. What is the value of a 4%, $100 bond, due in 3 years and 3 months, interest payable semiannually, plus accrued interest? The effective interest rate is 6%.

3. Find the purchase price of a 5% bond, interest payable annually. The bond is redeemable at par in 10 years and 4 months, and is bought on a 6% basis.

4. Find the cash price to be paid for a 6% bond, interest payable semiannually, and redeemable at par in 9 years and 9 months, bought on a 4% basis.

5. What is the value of five $1000, 8% bonds, interest payable semiannually, due in 3 years and 4 months, purchased on a 6% basis?

Bonds to be redeemed above par.—It is often necessary for many large companies to borrow money temporarily, and those who have money will not invest unless assured of a high rate of interest. Speculators will not invest in bonds that pay only an average rate, and industrial concerns will not issue long-term bonds at a high rate of interest. As a compromise, there has come into prominence a form of bond

which bears only an ordinary rate of interest, but which is redeemable (usually at the option of the maker) at a premium.

In order to find the value of a bond redeemable above par at the option of the maker before it is due, it is necessary to determine the value—

First, on the assumption that the bond will be redeemed at the optional redemption date, and at the optional price.

Second, on the assumption that the maker will not redeem the bond until maturity, and that it will then be redeemed at the par value.

After the two prices have been found, the purchaser should pay the lower of the two prices.

Example.—What price should an investor pay for a $1000, 6% semiannual interest bond due in 15 years, with the privilege of redemption in 10 years at 105, if purchased on a 5% semiannual basis?

Calculation on the assumption that the bond will be redeemed at the end of 10 years at 105.—

Algebraic Formula.

$$\text{Redeemable value}\left(\frac{1}{(1+i)^n}\right)+\text{Periodic cash interest}\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)=\text{Value of bond.}$$

Arithmetical Substitution.

$$1050\left(\frac{1}{(1.025)^{20}}\right)+30\left(\frac{1-\frac{1}{(1.025)^{20}}}{.025}\right)=\text{Value of bond.}$$

Solution.

Part 1.

$(1.025)^{20}$	=	1.638616,	compound amount of 1 at 2½% for 20 periods.
$1 \div 1.638616$	=	.61027,	present worth of 1 at 2½% for 20 periods.
$\$1050 \times .61027$	=	$640.78,	present value of redemption price of bond.

Part 2.

.61027	= Present value of 1 at 2½% for 20 periods.		
1. — .61027	=	.38973,	compound discount on 1 at 2½% for 20 periods.
$.38973 \div .025$	=	15.5892,	present value of annuity of 1.
$\$30 \times 15.5892$	=	$467.67,	present value of annuity of $30.

Part 3.

$640.78 + $467.67 = $1108.45, value based on optional redemption.

Calculation on the assumption that the bond will not be paid until due.—

Algebraic Formula.

$$\text{Par value}\left(\frac{1}{(1+i)^n}\right)+\text{Periodic cash interest}\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)=\text{Value of bond.}$$

Arithmetical Substitution.

$$1000\left(\frac{1}{(1.025)^{30}}\right)+30\left(\frac{1-\frac{1}{(1.025)^{30}}}{.025}\right)=\text{Value of bond.}$$

Solution.

Part 1.

$(1.025)^{30}$ = 2.097567, compound amount of 1 at 2½% for 30 periods.
1. ÷ 2.097567 = .476742, present value of 1 at 2½% for 30 periods.
$1000 × .476742 = $476.74, present value of par of bond.

Part 2.

.476742 = Present value of 1 at 2½% for 30 periods.
1. — .476742 = .523258, compound discount on 1 at 2½% for 30 periods.
.523258 ÷ .025 = 20.93029, present value of an annuity of 1.
$30 × 20.93029 = $627.91, present value of an annuity of $30.

Part 3.

$476.74 + $627.91 = $1104.65, price of bond based on par.

A comparison of the two results shows:

$1108.44, value based on 15 year redemption price.
$1104.65, value based on 20 year redemption price.

Therefore, the purchaser should pay the latter price, or $1104.65.

Problems.

1. On a 5% semiannual basis, what price should an investor pay for a $1000, 6% semiannual interest bond due in 10 years, if the company has the option of redemption at the end of 5 years at 104?

2. On a 5% basis, what price should be paid for five $1000, 6% semiannual interest bonds due in 20 years? The bonds contain a clause giving the maker the option of redemption at the end of 15 years by paying a bonus or premium of 10%.

3. A bond, bearing interest at 5% per annum, payable annually, and repayable in 5 years, with a bonus of 10%, is for sale. What price can a purchaser pay who desires to realize 6% on his investment? (V^5 at 6% = .7473.) Construct a table of verification.—*American Institute Examination.*

4. What could a purchaser who wished to realize 3% on his investment give for a bond for $10,000, which had 4 years to run, at 5% interest, payable yearly, and at the end of 4 years was payable with a bonus of 10%?

5. Give formula, solution, and verification for the following:

	Effective Interest	*Coupon Interest*	*Amount of Bond*	*Redeemable Value*	*Time to Run*	*Coupons Payable*
(*a*)	4%	6%	100	110	8 years	Annually
(*b*)	6%	8%	1000	105	6⅓ "	Annually
(*c*)	8%	6%	500	108	4⅙ "	Semiannually
(*d*)	8%	4%	500	110	5¼ "	Semiannually

Serial redemption bonds.—Many public and private corporations desire to pay off a portion of their bonds each year instead of setting up a sinking fund. Serial redemption bonds may be redeemed in equal or unequal periodic amounts.

If the bonds are not redeemed in equal periodic amounts, it is difficult to derive a formula or plan by which computations may be shortened or systematized. But if the redemptions are to be made in equal amounts and at regular periodic dates, formulas and solutions may be derived to find the value of such an issue.

For the purpose of arriving at the present value of bonds to be redeemed in a series, it is well to analyze the issue into its component parts, and calculate the value of each component part separately. The following example will illustrate this point:

Example.—What is the present value of a bond issue of $10,000 bearing 5% interest payable annually? These bonds are to be redeemed serially, $2000 each year. Money is worth 6%, effective interest.

ANALYSIS OF THE CALCULATION OF THE VALUE OF A SERIAL REDEMPTION BOND

	First Period	*Second Period*	*Third Period*	*Fourth Period*	*Fifth Period*	*Multiplied by Present Value of Annuity of 1*	*Present Value of Annuity of Principal*	*Present Value of a Series of Annuities of Interests*
Principal	$2000	$2000	$2000	$2000	$2000	$4.212363	$8424.73	
Interest payments	100					.94339		$ 94.34
" "	100	100				1.83339		183.34
" "	100	100	100			2.67301		267.30
" "	100	100	100	100		3.46511		346.51
" "	100	100	100	100	100	4.21236		421.24
							$8424.73	$1312.73

SUMMARY

Present value of annuity of $2000 at 6%	$8424.73
Present value of 5 annuities of $100 each (the number of rents vary from 1 to 5 as shown above) at 6%	1312.73
Present value of the serial redemption bonds	$9737.46

A study of the analysis of the calculation of the value of a series of interest bearing serial redemption bonds shows two general divisions:

1. The calculation of the present value of an annuity, the rents of which are the same as the bonds which are redeemed annually.

2. The calculation of the present value of a series of annuities, the rents of which are the same as the periodic interest payments on each series of bonds.

The first part needs no explanation, since it is similar to the calculation of the present worth of an ordinary annuity.

The sum of the values of the second part may be found by calculating the value of each separate annuity and adding the values found. The following is a short method of calculation:

Procedure, part 1.—(*a*) Find the present value of an annuity of 1 at the effective rate per cent for as many periods as the bond has serial payments.

(*b*) Multiply the present value of the annuity of 1 found in (*a*) by the number of dollars to be paid each period on the principal of the bonds.

(*c*) Divide the "annuity discount" found in (*b*) by the rate per cent expressed decimally, to find the average annuity price on 1.

(*d*) Multiply the average annuity price on 1 by the number of dollars in each interest payment or rent. The result will be the value of a series of annuities.

Algebraic Formula.

$$\left(\begin{matrix}\text{Annual}\\ \text{amount}\\ \text{deemed}\end{matrix}\right)\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)+\left(\begin{matrix}\text{Cash}\\ \text{interest}\end{matrix}\right)\left(\frac{\text{Number of annuities}-\frac{1-\frac{1}{(1+i)^n}}{i}}{i}\right)=\text{Present value.}$$

Arithmetical Substitution.

$$\$2000\left(\frac{1-\frac{1}{(1.06)^5}}{.06}\right)+\$100\left(\frac{5-\frac{1-\frac{1}{(1.06)^5}}{.06}}{.06}\right)=\text{Present value.}$$

Solution.

Part 1.

$(1.06)^5$	=	1.3382256, compound amount of 1 at 6% for 5 periods.
$1 \div 1.3382256$	=	.7472582, present worth of 1 at 6% for 5 periods.
$1 - .7472582$	=	.2527418, compound discount on 1 at 6% for 5 periods.
$.252742 \div .06$	=	4.2123638, present worth of annuity of 1 at 6% for 5 periods.
$\$2000 \times 4.2123638$	= $8424.73,	present worth of annuity of $2000.

Part 2.

4.2123638	= Present worth	of annuity of 1 at 6% for 5 periods.
$5. - 4.2123638$	= .7876362,	annuity discount on 5 annuities.
$.7876362 \div .06$	= 13.12727,	cumulative present worth of annuities.
$\$100 \times 13.1273$	= $1312.73,	cumulative present worth of annuity of $100.

Part 3.

$\$8424.73 + \$1312.73$	= $9737.46,	sum of present worth of annuities.

Problems.

1. State the purchase price of $5000 of serial bonds, issued January 1, 1923, with 4% interest, payable annually, and dated to mature in five equal annual installments. Money is worth 5%.

2. Verify the solution of the above problem by setting up a columnar table showing:

(a) Date of maturity.
(b) Bonds outstanding.
(c) Coupon interest each year.
(d) Effective interest each year.
(e) Accumulation of discount.

3. State the purchase price of $50,000 of serial bonds, issued January 1, 1920, bearing 6% interest, coupons due each year. These bonds are to mature serially in equal annual payments, beginning January 1, 1921, and each year thereafter for 10 years. Money is worth 4%.

4. Work out a columnar table for problem 3.

5. Set up a columnar table showing the requirements stated in problem 2 for a $20,000, 5%, semiannual serial

bond issue payable in 10 equal semiannual payments. Money is worth 6%, convertible semiannually.

Frequency of redemption periods.—In the foregoing serial redemption problem, a bond was redeemed at each interest date, but it is more usual to find the interest payable semiannually, while the bond redemptions occur once a year.

Example.—A $6000 issue of serial bonds is to be redeemed in equal installments of $2000, at the end of each year, for 3 years. The coupons are at the rate of 5%, payable semiannually. If money is worth 6%, convertible semiannually, what is the present value of the bond? (For the sake of illustration, short-term bonds are used.)

ANALYSIS OF THE CALCULATION OF THE VALUE OF A SERIAL REDEMPTION BOND AND INTEREST PAYMENTS WHEN INTEREST IS PAYABLE EACH SIX MONTHS

	Assumed Dates						*Present Value of an Annuity of 1 at 6.09%*	*Present Value of an Annuity of 1 at 3%*	
	June 30	*Dec. 31*	*June 30*	*Dec. 31*	*June 30*	*Dec. 31*			
Principal of Bond		$2000		$2000		$2000	$2.66857		$5337.14
" "	$25			25		25	2.66857		66.71
Semiannual Interest		25						$ .97087	24.27
" "	25	25						1.91346	47.84
" "	25	25	$25					2.82861	70.71
" "	25	25	25	25				3.71709	92.93
" "	25	25	25	25	$25			4.57970	114.49
" "	25	25	25	25	25	25		5.41719	135.43
									$ 485.67

SUMMARY

Present value of annuity of $2000 at 6.09%	$5337.14
Present value of annuity of $25 at 6.09%	66.71
Present value of six annuities of $25 each at 3%, compounded semiannually	485.67
	$5889.52

A study of the analysis of the calculation of the value of a series of interest-bearing serial redemption bonds with interest payments oftener than the redemption of bonds shows three divisions:

1. The calculation to find the present value of an annuity of the $2000 annual payment on the bond at the effective annual rate of 6.09%.

2. The calculation to find the present value of an annuity of one $25 interest payment due at the end of each year.

3. The calculation to find the present value of the series of six annuities of $25 each at 3% semiannual interest.

It should be noticed that one interest payment of $25 due at the end of each year is separated from the other interest payments. This separation is made to reduce the remaining interest payments to a series of annuities of regularly increasing terms.

The computation may be shortened by combining the annual bond redemption payment of $2000 and the annual $25 interest payment, and then finding the present value of an annuity, the rents of which would be $2025 at 6.09% per annum. The formula and solution are worked by a short method similar to the one explained on page 291.

Algebraic Formula.

$$\left.\begin{matrix}\text{Annual}\\ \text{amount}\\ \text{redeemed}\\ \text{plus 1}\\ \text{coupon}\end{matrix}\right)\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)+\left(\begin{matrix}\text{Cash}\\ \text{interest}\end{matrix}\right)\left(\frac{\text{Number of annuities}-\frac{1-\frac{1}{(1+i)^n}}{i}}{i}\right)=\text{Present value.}$$

Arithmetical Substitution.

$$\$2025\left(\frac{1-\frac{1}{(1.0609)^3}}{.0609}\right)+25\left(\frac{6-\frac{1-\frac{1}{(1.03)^6}}{.03}}{.03}\right)=\text{Present value.}$$

Solution.

Part 1.

$(1.03)^2$	=	1.0609,	effective ratio of increase.
$1.0609 - 1$	=	.0609,	effective rate per annum.
$(1.0609)^3$	=	1.19405228,	compound amount of 1 at 6.09% for 3 periods.
$1 \div 1.19405228$	=	.8374843,	present worth of 1 at 6.09% for 3 periods.
$1 - .8374843$	=	.1625157,	compound discount on 1 at 6.09% for 3 periods.
$.1625157 \div .0609$	=	2.668568,	present value of an annuity of 1 at 6.09% for 3 periods.
$\$2025 \times 2.668568$	=	\$5403.85,	present value of annuity of \$2025 at 6.09% for 3 periods.

Part 2.

$(1.03)^6$	=	1.19405228,	compound amount of 1 at 3% for 6 periods.
$1 \div 1.19405228$	=	.8374843,	present worth of 1 at 3% for 6 periods.
$1 - .8374843$	=	.1625157,	compound discount on 1 at 3% for 6 periods.
$.1625157 \div .03$	=	5.4171914,	present value of an annuity of 1 at 3% for 6 periods.
$6. - 5.4171914$	=	.5828086,	annuity discount on 6 annuities.
$.5828086 \div .03$	=	19.42695,	present value of 6 annuities of 1 at 3% for 6 periods.
$\$25 \times 19.42695$	=	\$485.67,	present value of 6 annuities of \$25 each.

Part 3.

$\$5403.85 + \$485.67$	=	\$5889.52,	value of serial bond.

Serial redemption bonds.—If the bonds are to be redeemed in any other than equal annual payments beginning at the end of the first year, an analysis must be made of the component parts and calculations made of each part separately.

Example.—What is the value of an issue of 5% serial redemption bonds for \$10,000, \$2,000 to be redeemed at the end of six years and \$2,000 at the end of each year until all of the bonds are redeemed? Interest coupons are payable semiannually. Money is worth 6% converted semiannually.

CHART ANALYSIS OF A SERIES OF SERIAL REDEMPTION BONDS AND INTEREST PAYMENTS

	Semiannual Periods																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
												2000		2000		2000		2000		2000
1st	25	25	25	25	25	25	25	25	25	25	25	25								
Series	25	25	25	25	25	25	25	25	25	25	25	25								
2nd	25	25	25	25	25	25	25	25	25	25	25	25	25	25						
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25						
3rd	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25				
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25				
4th	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25		
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25		
5th	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Series	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Totals	250	250	250	250	250	250	250	250	250	250										

From the chart analysis of the example there are found to be four component parts.

1. An annuity of $2000, the payments of which are to be made at the end of each year for 5 years, at an effective interest rate of 6% convertible semiannually, or 6.09% annually. The series of annuities is deferred for 5 years.

2. An annuity, the rents of which are $25, payable at the end of the sixth year and annually thereafter at 6.09%.

3. A series of 10 annuities of unequal length, the rents of which are $25, payable at the end of each half year. Each annuity is for 1 period longer than the preceding one. The rate of effective interest is 3%. This series of annuities is deferred for 5 years.

4. A series of 10 annuities of equal length, the rents of which are $25, payable at the end of each period at 3%.

Formulas will not be given for this problem, as it does not involve anything new, but only the combining of certain principles already described.

To shorten the calculation, (1) and (2) above can be combined, making a deferred annuity the rents of which are $2025 for 5 periods at 6.09%.

Procedure.—(*a*) Find the present value at the beginning of the sixth year of (1) and (2) combined.

(*b*) Find the present value of (3) at the beginning of the sixth year.

(*c*) Multiply the sum of the present values found in (*a*) and (*b*) by the present worth of 1 for 10 periods at 3%. The result is the present value at the beginning of the first period of all the bonds and interest payments falling due after the end of the fifth year.

(*d*) Calculate the present value of (4); this result is the value at the beginning of the first year.

(*e*) Add the results found in (*c*) and (*d*) and the sum is the present value of a series of serial redemption bonds and all interest payments.

Solution.

Note.—The compound amount and present value of $(1.0609)^5$ is the same as for $(1.03)^{10}$.

Part 1.

$(1.0609)^5$ = 1.3439164, compound amount of 1 for 5 periods at 6.09%.
1. ÷ 1.3439164 = .7440939, present value of 1 for 5 periods at 6.09%.
1. — .7440939 = .2559061, compound discount on 1 for 5 periods at 6.09%.
.2559061 ÷ .0609 = 4.2020707, present value of an annuity of 1 for 5 periods at 6.09%.
$2025 × 4.2020707 = $8509.19, value of the annuity of $2025 for 5 periods at 6.09%.

Part 2.

.7440939, present value of 1 for 10 periods at 3% (same as for 1 for 5 periods at 6.09%).
1. — .7440939 = .2559061, compound discount on 1 for 10 periods at 3%.
.2559061 ÷ .03 = 8.530202, present value of an annuity of 1 for 10 periods at 3%.
10. — 8.530202 = 1.469798, annuity discount on the series of 10 annuities of 1.
1.469798 ÷ .03 = 48.9932, present value of the series of 10 annuities of 1.
$25 × 48.9932 = $1224.83, present value of the series of 10 annuities of $25 each.

Part 3.

$8509.19 + $1224.83 = $9734.02, sum of present value at the beginning of the sixth year.
.7440939, present value of 1 for 10 periods at 3%.
$9734.02 × .7440939 = 7243.02, present value serial bonds and the series of the 10 annuities.

Part 4.

8.530202, present value of an annuity of 1 for 10 periods at 3%.
$25 × 10 = $250, sum of the 10 payments at the end of each period.
$250 × 8.530202 = $2132.55, present value of the annuity of 250 for 10 periods at 3%.

Part 5.

$7243.02 + $2132.55 = $9375.57, present value of serial bonds and all interest payments.

VERIFICATION OF CALCULATION OF THE VALUE OF SERIAL REDEMPTION BONDS

End of Period	*Bonds Redeemed*	*Effective Interest*	*Coupon Interest*	*Amortization of Discount*	*Bonds Less Discount*
					$9375.57
1.....		$281.27	$250.00	$31.27	9406.84
2.....		282.20	250.00	32.20	9439.04
3.....		283.17	250.00	33.17	9472.21
4.....		284.17	250.00	34.17	9506.38
5.....		285.19	250.00	35.19	9541.57
6.....		286.25	250.00	36.25	9577.82
7.....		287.33	250.00	37.33	9615.15
8.....		288.45	250.00	38.46	9653.60
9.....		289.61	250.00	39.61	9693.21
10.....		290.80	250.00	40.80	9734.01
11.....		292.02	250.00	42.02	9776.03
12.....	$2000.00	293.28	250.00	43.28	7819.31
13.....		234.58	200.00	34.58	7854.89
14.....	2000.00	235.62	200.00	35.62	5889.51
15.....		176.69	150.00	26.69	5916.20
16.....	2000.00	177.49	150.00	27.49	3943.69
17.....		118.31	100.00	18.31	3962.00
18.....	2000.00	118.86	100.00	18.86	1980.86
19.....		59.43	150.00	9.43	1990.29
20.....	2000.00	59.71	150.00	9.71	.00

Problems.

1. The State Highway Department of Minnesota desires to know the value of road bonds to be issued; par value $100,000, bearing 4% semiannual interest, and to be redeemed serially in installments of $20,000 each, the first payment of which is to be made one year from issue, the other payments to be made annually thereafter. Money is worth 6%, compounded semiannually.

Draw up a table of analysis for the above problem as a proof of your answer.

2. A corporation wishes to float an issue of serial bonds for $50,000. These bonds are to be redeemed in yearly installments of $10,000 each, beginning with the sixth year. The interest rate is 6%, payable semiannually; the effective rate is 5%, convertible semiannually.

Draw up a table of analysis for the above.

From the analysis, draw up the formulas and solution.

3. At what price must $50,000 worth of bonds be purchased to net the purchaser 6% annual effective interest, if the bonds bear 5% interest coupons, payable semiannually?

The bonds are to be redeemed serially, the first $5000 to be redeemed at the end of the fifth year, and $5000 each year thereafter until all the bonds are redeemed.

4. Set up a schedule in columnar form showing the book value, cash interest, effective interest, amortization of premium, and par value of bonds outstanding each year on an issue of bonds of $40,000, dated January 1, 1923, bearing 3% semiannual cash coupons, sold to produce 5% effective interest, convertible semiannually.

These bonds are to be retired as follows:

$10,000	on	January	1,	1928,	at 104
$10,000	"	"	"	1929,	" 103
$10,000	"	"	"	1930,	" 102
$10,000	"	"	"	1931,	" 101

Bonds redeemable from a fund.—Many bonds are redeemed periodically from a fund; that is, as soon as money is put into the fund or at the end of an interest period, it is used to redeem outstanding bonds. No difficulty would be encountered, except for the fact that bonds are usually issued in denominations of $100, $500, or $1000, and the payments into the fund may exceed the expenditures from the fund for the redemption of bonds, thus leaving a residue in the fund which should earn interest.

Example.—Show the periodic payment into a sinking fund, the interest accrued periodically on bonds redeemed, total amount to invest in sinking fund each period, amount of bonds purchased periodically for the sinking fund, the cash balance held in the treasury, and the interest on the cash balance held in the treasury. The issue of bonds is $100,000 (par $100), and the bond interest and sinking fund interest is 6%. The bonds are to run for 5 years (interest payable semiannually), and are to be kept alive in the treasury. Sinking fund payments are to be made semiannually. The interest is to be paid by the trustee on the balance remaining in the fund.

Let i = the effective interest per period.
n = the number of periods.

Algebraic Formula. *Arithmetical Substitution.*

$$\frac{P}{\dfrac{1-\dfrac{1}{(1+i)^n}}{i}} = R. \qquad \frac{100{,}000}{\dfrac{1-\dfrac{1}{(1.03)^{10}}}{.03}} = \text{Rent.}$$

Solution.

$(1.03)^{10}$ = 1.34391638, compound amount of 1 for 10 periods at 3%.
$1 \div 1.34391638$ = .74409391, present worth of 1 for 10 periods at 3%.
$1 - .74409391$ = .25590609, compound discount on 1 for 10 periods at 3%.
$.25590609 \div .03$ = 8.5302028, present worth of an annuity of 1 for 10 periods at 3%.
$\$100{,}000 \div 8.5302028$ = $11,723.05, annual payment.

The remaining part of the solution can be seen from the following table:

COLUMNAR TABLE

Periodic Payment	*Balance From Last Period*	*Total Amount Available*	*Bond Interest*	*Bonds Redeemed*	*Cash Balance*	*Interest on Cash Balance*	*Bonds Outstanding*
							$100,000
$11,723.05		$11,723.05	$3,000	$ 8,700	$23.05	$ .69	91,300
11,723.05	$23.74	11,746.79	2,739	9,000	7.79	.23	82,300
11,723.05	8.02	11,731.07	2,469	9,200	62.07	1.86	73,100
11,723.05	63.93	11,786.98	2,193	9,500	93.98	2.82	63,600
11,723.05	96.80	11,819.85	1,908	9,900	11.85	.36	53,700
11,723.05	12.21	11,735.26	1,611	10,100	24.26	.73	43,600
11,723.05	24.99	11,748.04	1,308	10,400	40.04	1.20	33,200
11,723.05	41.24	11,764.29	996	10,700	68.29	2.05	22,500
11,723.05	70.34	11,793.39	675	11,100	18.39	.55	11,400
11,723.06	18.94	11,742.00	342	11,400			

In the above problem, the first payment into the fund is $11,723.05. The payment on the interest will be $3000, leaving $8723.05 to be used for the redemption of bonds, and as the bonds issued in this problem are of the denomination of $100, only $8700 of this fund can be used for the redemption of bonds, leaving a balance of $23.05 cash in the hands of the trustee. Even this would cause no difficulty, except for the fact that it is not customary for the balance in cash left in the hands of the trustee to draw interest.

It is necessary in order to verify the solution of a problem of this kind to charge the trustee with interest at the sinking fund rate on the balance remaining in his hands. The interest on the $23.05, the cash balance in the hands of the trustee, for 6 months at 3% is $.69. This interest added to the cash balance gives the balance from the last period.

Problems.

1. Prepare a table showing the semiannual periodic payments to the sinking fund, interest accrued on bonds, par value of bonds purchased semiannually, cash balance in the hands of the trustee, and interest on cash balance, where the issue of bonds is $100,000, interest at 5% payable semiannually for 8 years, issue to be purchased by the trustee at par and kept alive in the treasury. Money is worth 5% semiannually. Par value of bonds is $1000.

2. Prepare a table as above for an issue of $500,000, 6%, 10-year bonds floated by a corporation, coupons payable semiannually. Bonds are to be redeemed semiannually by the trustee into whose hands are to be placed the semiannual sinking fund payments necessary for redemption of principal and payment of interest. Par value of bond is $1000, and the sinking fund rate is 6%.

Effective rate of interest on bonds.—One very important question to the investor is, "What rate of interest will I receive on this bond?" or "What is the effective rate on the money invested?" The business man knows the amount of each interest coupon, but the coupon rate is based on par value and not on the amount of money invested.

Because of the fact that the unknown effective interest rate must be used more than once in the algebraic formula for the calculation of the price of a bond, it is difficult to find the absolute effective rate. Two methods of finding an approximate effective rate may be used; one is that of interpolation in conjunction with a bond table, or by test rates used with formulas for finding the value of bonds; the other is by formulas specially constructed to give approximately the rate required. Most of these formulas will give a rate accurate enough for any ordinary accounting problem.

Effective rate of bonds at a premium.—

Example.—If a $100, 3-year, 6% bond, bearing semiannual interest coupons, is bought at $105.38, what rate of interest will be realized on the investment?

SECTION OF BOND TABLE

CASH INTEREST PAYABLE SEMIANNUALLY

Effective Interest Rate	*3%*	*3½%*	*4%*	*4½%*	*5%*	*6%*	*7%*
3.75	$97.89	$99.30	$100.70	$102.11	$103.52	$106.33	$109.14
3.80	97.75	99.16	100.56	101.97	103.37	106.18	108.99
3.875	97.54	98.95	100.35	101.75	103.16	105.96	108.77
3.90	97.48	98.88	100.28	101.68	103.09	105.89	108.70
4.00	97.20	98.60	100.00	101.40	102.80	105.60	108.40
4.10	96.92	98.32	99.72	101.12	102.52	105.31	108.11
4.12½	96.86	98.25	99.65	101.05	102.45	105.24	108.04
4.20	96.65	98.05	99.44	100.84	102.23	105.02	107.82
4.25	96.51	97.91	99.30	100.70	102.09	104.88	107.67

Solution.

In the 6% column of the section of the bond table given above will be found opposite 4% the price of $105.60, and opposite the 4.1% the price of $105.31. The rate is therefore somewhere between 4% and 4.1%. A more exact rate may be found, as follows:

INTERPOLATION

Value on a basis of 4%	$105.60
Value on a basis of 4.1%	105.31
Difference in value caused by a difference in rate of .1%	.29
Value on a basis of 4%	105.60
Price paid	105.38
Difference between price paid and value on a basis of 4%	.22

If the difference between the price paid and the value on a basis of 4% is $.22, the difference in the value caused by a difference of .1% in the rate is $.29, the rate earned will be $^{22}/_{29}$ of .1% greater than 4%, or 4.076%.

Effective rate of bonds at a discount.—

Example.—What should be the rate of income on an investment in a 4%, semiannual, 3-year bond bought at $99.35?

Solution.

By reference to the table above, it is found that the	
Value of a 4% bond at 4.2% effective is:	$99.44
Value of a 4% bond at 4.25% effective is:	99.30
Difference in value caused by a difference in a rate of .05% is equivalent to	.14
Value on a basis of 4.2% is	$99.44
Price of bond is	99.35
Difference is	.09
$9/14$ of .05%	.032%
4.20% + .032%	4.232%

$\therefore$ 4.232% = the approximate rate.

If no bond table is available.—If no bond table is available, an approximate rate may be computed by using test rates, but care must be exercised so that the rates used are as close to the actual rate as it is possible to estimate.

Example.—If a $100, 5%, 5-year bond, interest payable semiannually, is bought at $97.31, what rate of interest will be realized on the investment?

Solution.

The first step is to find the cost at an estimated effective rate. Assume this rate to be 5½%. By the formula, previously given, for finding the purchase price of bonds, this bond at 5.5% is worth $97.84. As this price is higher than the price paid, the rate is too low, and it is necessary to try a higher rate. Make a second trial at 5.75%. The purchase price is then found to be $96.78. As one price is slightly above and the other slightly below the actual price paid, the approximate rate is found by interpolation.

INTERPOLATION

Value on a 5.50% basis	$97.84
Value on a 5.75% basis	96.78
Difference in value caused by a difference in a rate of ¼%	1.06
Value on a 5.50% basis	97.84
Price paid for bond	97.31
Difference between price at 5.50% and price paid	$ .53

If the difference between the price paid and the price at 5.50% is $.53, and the difference between the value on a

5.50% basis and the value on a 5.75% basis is $1.06, the price paid will be approximately .53/1.06 of the difference between 5.50% and 5.75%, added to 5.50%, or 5.625%.

Problems.

1. Find the effective rate in the following:

	Purchase Price	*Nominal Rate*	*Par Value*	*Time in Years*	*Interest Payable*
(a)	$100.25	4%	$100	3	Semiannually
(b)	96.70	3%	100	3	"
(c)	108.50	7%	100	3	"
(d)	106.30	6%	100	3	"
(e)	96.94	5%	100	10	"
(f)	91.47	4%	100	5	"
(g)	88.25	3%	100	25	Annually
(h)	132.00	6%	100	20	Semiannually
(i)	1120.00	5%	1000	14	"
(j)	475.00	4%	500	5	"
(k)	450.00	4%	500	10	"

2. The 4¾% Victory notes mature at par on May 20, 1923. If a purchaser buys at $96.20 on May 20, 1920, calculate the approximate yield per cent.

Given:	2¾%	3%
A_6	5.4624	5.4172
V_6	.8498	.8375

—*American Institute Examination.*

3. Verify the correctness of the following rate of yield:

$10,000,000

GENERAL PETROLEUM CORPORATION

FIVE-YEAR 6% GOLD NOTES

Due April 15, 1928. To be dated April 15, 1923.

Authorized and presently to be issued $10,000,000.

Interest payable April 15 and October 15, without deduction for normal Federal Income Tax not in excess of 2%. Principal and interest payable in San Francisco and New York. Coupon notes in denominations of $1000 and $500. Redeemable, at the option of the Company, at any time, on 60 days' notice, at 102½ on or before October 15, 1923, the premium thereafter decreasing ¼ of 1% for each six months' period or fraction thereof.

BANK OF CALIFORNIA, N. A. SAN FRANCISCO, TRUSTEE.

Price 97½ and interest to yield about 6.60% to maturity.

4. Verify the correctness of the following yield:

$1,000,000

IOWA SOUTHERN UTILITIES COMPANY

First and Refunding Mortgage 6% Gold Bonds, Series of 1923

Dated February 1, 1923. Due February 1, 1943.

Listed on the Chicago Stock Exchange.

Interest payable February 1 and August 1. Coupon bonds in $1000, $500, and $100 denominations, registerable as to principal only. Redeemable on any interest date at 107½ to January 1, 1928; 105 to July 1, 1933; and 102½ thereafter. Principal and interest payable in New York and Chicago, without deduction for any Federal Income Tax not to exceed 2%.

Price 93 and interest to yield 6⅝%.

Amortization of discount and expense on serial redemption bonds.—The calculation of the amortization of bond discount, bond premium, or bond discount and expense, when bonds are to be redeemed serially, or in unequal amounts, is a problem which requires particular attention to make an equitable distribution over the period of years. The two methods commonly used are:

1. The amortization of the discount or premium by the sum-of-periods method.
2. The amortization of the discount or premium by the scientific or effective-interest method.

Sum-of-periods method.—It would be incorrect to write off the discount or premium, or the discount and expense, on serial redemption bonds or on a bond issue which has no regular redemption period, by the straight-line method. In some cases it would be impossible to calculate the portion to be amortized by the scientific method. Because of the ease of calculations and the fair degree of accuracy, the sum-of-periods method is commonly used.

Procedure.—(*a*) Find the sum of the totals of the bonds outstanding each period during the life of the bonds.

(*b*) If the sum found in (*a*) is used as the denominator of the fraction and the total bonds outstanding for any one period as the numerator, the fraction will be the portion of the bond discount or premium, or bond discount and expense, to be written off for the period.

Example.—How should the discount be amortized equitably over the life of bonds in an issue of ten bonds of $1000 each, bearing 5% interest? These bonds are to be redeemed, $3000 at the end of the sixth year, $3000 at the end of the eighth year, and $2000 at the end of each year thereafter until paid. The bonds were sold at 96.

Solution.

Periods of ½ Year Each	*Bonds Outstanding*	*Fraction*	*Amortization Written Off*	*Discount on Bonds*	*Bonds Less Discount*
	$10,000.00			$400.00	$ 9,600.00
1	10,000.00	10/160	$25.00	375.00	9,625.00
2	10,000.00	10/160	25.00	350.00	9,650.00
3	10,000.00	10/160	25.00	325.00	9,675.00
4	10,000.00	10/160	25.00	300.00	9,700.00
5	10,000.00	10/160	25.00	275.00	9,725.00
6	10,000.00	10/160	25.00	250.00	9,750.00
7	10,000.00	10/160	25.00	225.00	9,775.00
8	10,000.00	10/160	25.00	200.00	9,800.00
9	10,000.00	10/160	25.00	175.00	9,825.00
10	10,000.00	10/160	25.00	150.00	9,850.00
11	10,000.00	10/160	25.00	125.00	9,875.00
12	7,000.00	10/160	25.00	100.00	6,900.00
13	7,000.00	7/160	17.50	82.50	6,917.50
14	7,000.00	7/160	17.50	65.00	6,935.00
15	7,000.00	7/160	17.50	47.50	6,952.50
16	4,000.00	7/160	17.50	30.00	3,970.00
17	4,000.00	4/160	10.00	20.00	3,980.00
18	2,000.00	4/160	10.00	10.00	1,990.00
19	2,000.00	2/160	5.00	5.00	1,995.00
20	000.00	2/160	5.00	0.00	000.00
		160/160	400.00		

Amortization of discount by the scientific method.—To find the amount of discount or premium on an issue of serial redemption bonds to be amortized by the scientific method, it is necessary to find the approximate effective rate of interest these bonds will bear. To find the approximate effective rate it is necessary to use other approximations.

Procedure.—(*a*) Determine the average life of the bonds in periods.

(*b*) Determine, by the use of a bond table or by annuity calculations, the approximate effective interest or rate for one bond having a life of the same number of periods as the average life found in (*a*).

(*c*) From a bond table or by annuity calculation, find

the values of one of the bonds which matures each year, at the approximate effective rate found in (*b*).

(*d*) Find the total value of all the bonds which mature each year by using the values found in (*c*).

(*e*) Add the products found in (*d*).

(*f*) Compare, by means of an approximate rate per cent, the sum found in (*e*) with the actual price received for the bonds to find the error.

(*g*) By the same process determine another approximate rate.

(*h*) By interpolation, determine the error by using approximate rates found above.

Using the example given under the sum-of-digits method, the solution would be:

Bonds	*Maturing in Years*	*Equivalent to One Bond Maturing in Years*
3	6	18
3	8	24
2	9	18
2	10	20
10		80

80 ÷ 10 = 8, or an average life of eight years.
9600 ÷ 100 = 96, the average sales price per hundred.

By reference to a bond table under 5%, semiannual interest for a bond maturing in 8 years, the nearest price to 96 is found to be 96.02. This amount is found opposite the effective rate per annum of 5⅝%.

The next step is to find the value of all the bonds of the series at 5⅝%, which is done by reference to the bond table under the cash rate of 5% and effective rate of 5⅝%.

The results will be:

Par Value of Bonds	*Years to Run*	*Value at 5⅝%*	*Total Value*
$3000.00	6	$96.85	$2905.50
3000.00	8	96.02	2880.60
2000.00	9	95.63	1912.60
2000.00	10	95.27	1905.40
Value of series			$9604.10

As sale price of $9600 for the bonds is below the price if sold at 5⅝% the effective rate is a little larger. A test at 5¾% would result in the following:

Par Value of Bonds	*Years to Run*	*Value of 5¾%*	*Total Value*
$3000.00	6	$96.24	$2887.20
3000.00	8	95.24	2857.20
2000.00	9	94.79	1895.80
2000.00	10	94.36	1887.20
Value of series			$9527.40

INTERPOLATION

Value of series at 5⅝%	$9604.10
" " " " 5¾%	9527.40
Difference caused by a difference in rate of ⅛%	76.70

Value of series at 5⅝%	$9604.10
Sales price	9600.00
Difference	4.10
4.10/76.70 of ⅛%	.006682
Estimated rate of	5.625
Add fractional rate found	.006682
Effective rate	5.631682
Effective rate to be used semiannually (divide by 2)	2.815841%

TABLE SHOWING AMORTIZATION OF DISCOUNT FOR SERIAL REDEMPTION BONDS

End of Period	*Bonds Redeemed*	*Effective Rule 2.815841%*	*Coupon Rate 2.5%*	*Amortization of Discount*	*Bonds Less Discount*
					$9600.00
1		$270.32	$250.00	$20.32	9620.32
2		270.89	250.00	20.89	9641.21
3		271.48	250.00	21.48	9662.69
4		272.09	250.00	22.09	9684.78
5		272.71	250.00	22.71	9707.49
6		273.35	250.00	23.35	9730.84
7		274.01	250.00	24.01	9754.85
8		274.68	250.00	24.68	9779.53
9		275.38	250.00	25.38	9804.91
10		276.09	250.00	26.09	9831.00
11		276.83	250.00	26.83	9857.83
12	$3000.00	277.58	250.00	27.58	6885.41
13		193.88	175.00	18.88	6904.29
14		194.41	175.00	19.41	6923.70
15		194.96	175.00	19.96	9643.66
16	3000.00	195.52	175.00	20.52	3964.18
17		111.63	100.00	11.63	3975.81
18	2000.00	111.95	100.00	11.95	1987.76
19		55.97	50.00	5.97	1993.73
20	2000.00	56.14	50.00	6.14	.13 *

* Error caused by approximation and by the use of bond tables having only 4 places.

Problems.

1. The *A B* Company issued $500,000 of first mortgage bonds, bearing 5% semiannual interest which it sold for $480,000. These bonds are to be retired as follows:

$	50,000	at	the	end	of	the	third year
	50,000	"	"	"	"	"	fourth "
	75,000	"	"	"	"	"	fifth "
	75,000	"	"	"	"	"	sixth "
	75,000	"	"	"	"	"	seventh "
	75,000	"	"	"	"	"	eighth "
	100,000	"	"	"	"	"	ninth "

Find a solution and prepare schedules by the sum-of-periods method.

2. Find a solution and prepare schedules by the scientific method for the above problem.

Chapter 31

ASSET VALUATION ACCOUNTS

Asset valuation.—At the moment an asset is purchased for use it is said to be worth cost. When this asset can no longer be used for the purpose for which it was purchased, and can be sold for little or nothing, it is said to be worth scrap value. The difference between the scrap value and cost is the depreciation or depletion.

Depreciation.—Depreciation is the deterioration caused by wear, tear, action of the elements, obsolescence, supersession, and inadequacy.

Depletion.—Depletion is the extinction of an asset by a reduction in the quantity.

Depreciation methods.—There are many methods of calculation of the periodic depreciation charge. Probably the most valuable ones are the following:

1. Straight-line method.
2. Working-hours or unit-product method.
3. Sum-of-digits method.
4. Sinking-fund method.
5. Annuity method.
6. Fixed percentage of diminishing value method.

Each of the many methods of calculation of depreciation has its valuable points. Some are based on scientific principles, while others are used because of their ease of calculation.

The accountant should know how to calculate depreciation by each method and should be able to discuss the advantages or disadvantages of each, and be able to formulate tables of comparisons showing the results of each.

Straight-line method.—This is the most common method and the simplest of computation.

Procedure.—(*a*) Find the difference between the cost and the scrap value.

(*b*) Divide the difference found in (*a*) by the number of periods the asset is expected to be of service. The result is the depreciation charge per period.

Example.—What will be the depreciation charge and asset valuation at the end of each year for an asset costing $1000, with an estimated life of ten years, and having an estimated scrap value of $100?

Formula.

Cost — Scrap = Depreciation.
Depreciation ÷ Number of years = Annual charge.

Arithmetical Substitution.

$1000 — $100 = $900.
900 ÷ 10 = 90.

TABLE

Years	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
			$1000
1	$90	$ 90	910
2	90	180	820
3	90	270	730
4	90	360	640
5	90	450	550
6	90	540	460
7	90	630	370
8	90	720	280
9	90	810	190
10	90	900	100

Problems.

1. Set up a table showing the annual depreciation, carrying value, and accumulated depreciation of an asset costing $2500, having a life of 10 years, and having a scrap value of $300.

2. Show by tables, the information necessary to present to a business man the results of the straight-line depreciation of his factory machines as follows:

	Cost	*Scrap Value*	*Life*
Lathes	$5000	$500	10 years
Milling Machine	3000	200	12 "
Power Equipment	4000	200	10 "
Furniture	600	50	15 "

Working-hours or unit-product method.—This method is based on the number of hours the machine operates, or the number of units produced. It is especially useful in a factory which must operate on an unusual basis of long hours or high pressure. If a machine which, under ordinary conditions could be operated for 10 years or approximately 24,000 hours, is, owing to business conditions, operated during the first year for 5000 hours, the depreciation should, of course, be written off in a shorter time than 10 years.

Example.—A certain one-purpose machine, which cost $1000 (no scrap value) has been installed in a factory. Machines of this class have a record of producing 10,000 units of product each, during their life. Assuming the annual production is as given below, set up tables showing the depreciation to be written off each year.

Year	*Units Produced*
1	1,000
2	2,000
3	1,800
4	1,000
5	1,000
6	1,200
7	1,200
8	800
	10,000

Solution.

TABLE

Year	*Unit Fraction of Cost*	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
				$1000
1	1000/10000	$100	$100	900
2	2000/10000	200	300	700
3	1800/10000	180	480	520
4	1000/10000	100	580	420
5	1000/10000	100	680	320
6	1200/10000	120	800	200
7	1200/10000	120	920	80
8	800/10000	80	1000	0

Problem.

1. Show in appropriate form, the yearly depreciation, accumulated depreciation, and asset value of a machine which cost $5000, and which will have a scrap value of $200. As-

sume that machines of this class have a working-hour average life of 24,000; also assume that the machine will be run as follows:

Year	*Working Hours*
1	2000
2	2000
3	1800
4	2600
5	2800
6	2000
7	3000
8	3000
9	3000
10	1800

Sum-of-digits method.—This method is easy to calculate if understood. For those who believe the depreciation charge should be large during the first years of the useful life of the asset, the sum-of-digits method will be found useful.

Procedure.—(*a*) Find the sum of the digits, or numbers representing the periods of useful life of the asset. Use this sum as a denominator for certain fractions.

(*b*) Use the same digits or numbers in inverse order as the numerators of these fractions.

(*c*) Compute the periodic depreciation by multiplying the total depreciation by the fractions obtained in (*a*) and (*b*).

Example.—What would be the depreciation charges for an asset valued at $1000, scrap value of $100, to be written down in nine years by the sum-of-digits method?

Solution.

TABLE

Year	*Fractional Part*	*Periodic Depreciation Charge*	*Accumulated Depreciation Reserve*	*Asset Value*
				$1000
1	9/45	$180	$180	820
2	8/45	160	340	660
3	7/45	140	480	520
4	6/45	120	600	400
5	5/45	100	700	300
6	4/45	80	780	220
7	3/45	60	840	160
8	2/45	40	880	120
9	1/45	20	900	100
45	45/45	900		

Note.—The common denominator of the fractions used in the second column is found by adding the first column. The numerators, it will be noticed, are the inverse of the number of years used in the first column.

Problem.

1. What would be the depreciation charges, the asset valuation, and the depreciation accumulation for each year on each of the following machines by the sum-of-digits method?

	Cost	*Scrap*	*Life of Asset*
Power Lathe	$1200	$200	10 years
Hack Saw	350	50	12 "
Turret Lathe	2500	400	7 "
Boiler	2600	300	8 "
Power Equipment	800	100	5 "
Delivery Equipment	2000	400	5 "

Present the information by means of tables.

Sinking-fund method.—The theory of the sinking-fund method is that the money set aside in a fund should draw interest, this interest to be added to the fund at the end of each year. The total accumulation of the fund should be equal to the depreciation during the life of the asset. While in theory a fund is set aside equal to the total accumulation, usually in practice no such fund exists.

Procedure.—(*a*) Find the amount of the total depreciation of the asset by deducting the scrap value from the cost.

(*b*) Divide the total depreciation found in (*a*) by the amount of an ordinary annuity of 1 at the sinking fund interest rate, for the number of periods of the life of the asset in order to find the periodic sum to be placed in the sinking fund.

(*c*) Add to the periodic sum found in (*b*), a sum equal to the interest on the sinking fund for the period, to find the periodic charge to depreciation, and the credit to reserve for depreciation.

Example.—Determine the periodic depreciation charge by the sinking-fund method for an asset costing $1000, with a scrap value of $100 at the end of 10 years, on a 6% interest basis.

Algebraic Formula.

$$\frac{\frac{\text{Cost} - \text{Scrap}}{(1+i)^n - 1}}{i} = \text{Periodic deposit in sinking fund.}$$

Arithmetical Substitution.

$$\frac{\frac{1000 - 100}{(1.06)^{10} - 1}}{.06} = \text{Periodic deposit in sinking fund.}$$

Solution.

$1000 − $100 = $900, amount to be depreciated.
$(1.06)^{10}$ = 1.7908477, compound amount of 1 for 10 periods at 6%.
1.7908477 − 1 = .7908477, compound interest on 1 for 10 periods at 6%.
.7908477 ÷ .06 = 13.180795, amount of annuity of 1 for 10 periods at 6%.
$900 ÷ 13.180795 = $68.28, periodic deposit in sinking fund.

The formula and solution just shown will give only the first periodic charge to depreciation, as each periodic charge will be an increased amount equal to the sum of the first periodic payment plus the interest on the accumulated depreciation reserve. The second table following shows the periodic charges to depreciation and the credits to the reserve account for each of the 10 years.

It is not necessary to accumulate a sinking fund in order to use the sinking-fund method of depreciation. The depreciation entries are independent of the fund entries. If a sinking fund were accumulated for the above problem, the entries would be as shown by the first table following:

TABLE SHOWING ENTRIES TO THE SINKING FUND

Year	*Debit to Sinking Fund*	*Credit to Cash*	*Credit to Interest*	*Accumulation of Fund*
1	$ 68.28	$ 68.28		$ 68.28
2	72.38	68.28	$ 4.10	140.66
3	76.72	68.28	8.44	217.38
4	81.32	68.28	13.04	298.70
5	86.20	68.28	17.92	384.90
6	91.37	68.28	23.09	476.27
7	96.86	68.28	28.58	573.13
8	102.67	68.28	34.39	675.80
9	108.83	68.28	40.55	784.63
10	115.37	68.29	47.08	900.00
	900.00	682.81	217.19	

TABLE SHOWING DEPRECIATION ENTRIES BY THE SINKING FUND METHOD OF DEPRECIATION

End of Year	*Depreciation Charge and Reserve Credit*	*Accumulation Depreciation Reserve*	*Asset Value*
			$1000.00
1	$ 68.28	$ 68.28	931.72
2	72.38	140.66	859.34
3	76.72	217.38	782.62
4	81.32	298.70	701.30
5	86.20	384.90	615.10
6	91.37	476.27	523.73
7	96.86	573.13	426.87
8	102.67	675.80	324.20
9	108.83	784.63	215.37
10	115.37	900.00	100.00

Problem.

1. Set up sinking fund depreciation tables for a manufacturer who owns the following, money being worth 6%:

	Cost	*Scrap Value*	*Life*
Office Furniture	$ 500	$ 125	8 years
Factory Furniture	1,000	100	10 "
Machinery	15,000	3,000	8 "
Delivery Equipment ...	4,000	500	5 "

Annuity method of depreciation.—The theory applied in this method is that the depreciation charge should contain interest on the carrying value of the asset in addition to the amount credited to the reserve.

The investment in property is regarded, first, as the amount of scrap value which draws interest, and second, as an investment in an annuity to be reduced by equal periodical amounts. The sum of the interest on the scrap value plus the equal periodical reduction on the investment is the charge to depreciation, offset by a credit to interest computed on the diminishing value of the property, and to the reserve account for the balance. This charge to depreciation is the same each period during the life of the property. The theory of an investment in an annuity is that the annuity is to be reduced by equal periodic payments, and as the credits to interest will decrease, the credits to the reserve must correspondingly increase.

Procedure.—(*a*) Find the difference between the cost and the scrap value.

(*b*) Divide the difference found in (a) by the present value of an annuity of 1.

(*c*) Calculate the interest on the scrap value for one period at the given rate per cent.

(*d*) Determine the sum of (*b*) and (*c*). This sum will be the periodic charge to operation for the period.

Example.—What will be the annual charge to depreciation by the annuity method of depreciation, for an asset valued at $1000, with a scrap value of $100, to be written off in 10 years, computed on a 6% basis?

Algebraic Formula.

$$\left[\frac{\text{Cost} - \text{Scrap}}{1 - \frac{1}{(1+i)^n}}{i}\right] + (\text{Scrap} \times i) = \text{Periodic charge.}$$

Arithmetical Substitution.

$$\left[\frac{1000 - 100}{1 - \frac{1}{(1.06)^{10}}}{.06}\right] + (100 \times .06) = \text{Periodic charge.}$$

Solution.

Part 1.

$1000 − $100 = $900, sum to be depreciated.
$(1.06)^{10}$ = 1.7908477, compound amount of 1 for 10 periods at 6%.
1. ÷ 1.7908477 = .5583948, present value of 1 for 10 periods at 6%.
1. − .5583948 = .4416052, compound discount on 1 for 10 periods at 6%.
.4416052 ÷ .06 = 7.360087, present value of an annuity of 1 for 10 periods at 6%.
$900 ÷ 7.360087 = $122.28, rent of the present value of annuity.

Part 2.

$100 × .06 = $6.00, interest on scrap value.

Part 3.

$122.28 + $6. = $128.28, periodic charge to depreciation.

In the above problem the $900 represents the present value of the sum to be spread over the life of the asset, and the $100 represents the scrap. In the following tables, the

fifth column always contains the carrying value of the annuity plus $100. The two tables are given to show the similarity between an annuity in which an investment was made and equal annual rents withdrawn, and the annuity method of depreciation.

TABLE OF REDUCTION OF AN ANNUITY

End of Period	*Rents Withdrawn*	*Credit to Interest*	*Amortization of Investment*	*Present Value of Annuity Plus $100*
				$1000.00
1	$128.28	$60.00	$ 68.28	931.72
2	128.28	55.90	72.38	859.34
3	128.28	51.56	76.72	782.62
4	128.28	46.96	81.32	701.30
5	128.28	42.08	86.20	615.10
6	128.28	36.91	91.37	523.73
7	128.28	31.42	96.86	426.87
8	128.28	25.61	102.67	324.20
9	128.28	19.45	108.83	215.37
10	128.29	12.92	115.37	100.00
Totals.....	1282.81	382.81	900.00	

TABLE OF REDUCTION OF THE VALUE OF A MACHINE

End of Period	*Depreciation Charge*	*Credit to Interest*	*Credit to Reserve*	*Value of Machine*
				$1000.00
1	$128.28	$60.00	$ 68.28	931.72
2	128.28	55.90	72.38	859.34
3	128.28	51.56	76.72	782.62
4	128.28	46.96	81.32	701.30
5	128.28	42.08	86.20	615.10
6	128.28	36.91	91.37	523.73
7	128.28	31.42	96.86	426.87
8	128.28	25.61	102.67	324.20
9	128.28	19.45	108.83	215.37
10	128.29	12.92	115.37	100.00
Totals.....	1282.81	382.81	900.00	

Problem.

A Company, believing that the annuity method of depreciation is the correct one, desires you to construct tables for the following machines (one table for each):

		Cost	*Scrap*	*Life*	*Interest Rate*
1.	On Lathes	$5000	$ 500	10 years	6%
2.	On Milling Machines ...	4000	1000	8 years	5%
3.	On Grinders	1000	200	5 years	4%
4.	On Motors	2000	200	6 years	8%

Fixed percentage of diminishing value.—This method of calculation gives a large depreciation charge during the first part of the life of the asset. The charge constantly decreases each period.

The difficulty encountered is that of finding the rate per cent to be used in the calculation of the charge.

Procedure.—(*a*) Divide the scrap value by the cost.

(*b*) Extract the root, corresponding to the number of periods of depreciation to be taken on the life of the asset, of the quotient obtained in (*a*).

(*c*) Deduct from 1 the result obtained in (*b*) to find the rate per cent to be used.

(*d*) Multiply the net asset or carrying value of the asset at the beginning of each period by the rate found in (*c*), to obtain the correct depreciation charge for the period.

Example.—What will be the depreciation charges for an asset valued at $1000, with a scrap value of $100, to be written off in 10 years by the fixed percentage of diminishing value method?

FOR THE CALCULATION OF THE RATE.

Algebraic Formula.

$$1 - \sqrt[n]{\frac{\text{Scrap value}}{\text{Cost value}}} = r.$$

Arithmetical Substitution.

$$1 - \sqrt[10]{\frac{100}{1000}} = \text{Rate.}$$

Solution.

Part 1.

100 ÷ 1000	=	.1
Log .1	=	— 1.000000
Changed, — 1.000000	=	9.000000 — 10.
9.000000 — 10. ÷ 10	=	.900000 — 1.
Changed,	=	— 1.9000000
The antilog of — 1.9000000	=	.794328
1. — .794328	=	.205672, or 20.5672%.

Part 2.

$1000 × 20.567	= $205.67, first depreciation charge.
1000 — 205.67	= 794.33, new asset value.
794.33 × 20.567	= 163.37, second depreciation charge.

(This process is continued for each of the ten years.)

TABLE (RATE 20.567+%)

Year	Periodic Depreciation Charge	Accumulated Depreciation Reserve	Asset Value
			$1000.00
1	$205.67	$205.67	794.33
2	163.37	369.04	630.96
3	129.77	498.81	501.19
4	103.08	601.89	398.11
5	81.88	683.77	316.23
6	65.04	748.81	251.19
7	51.66	800.47	199.53
8	41.04	841.51	158.49
9	32.60	874.11	125.89
10	25.89	900.00	100.00

Problems.

1. Construct a comparative columnar table showing the periodic depreciation charges for the straight-line, sum-of-digits, sinking-fund, annuity, and fixed-percentage-of-diminishing-value methods of an asset costing $10,000, with a probable life of 8 years and a scrap value of $200. Money is worth 4% per annum.

2. Construct comparative columnar tables showing the carrying value and annual depreciation charge, for the straight-line, sum-of-digits, sinking-fund, annuity, and fixed-percentage-of-diminishing-value methods of depreciation on an asset costing $2000, with a life of 8 years, money being worth 5%. The asset has no scrap value, but for the purpose of calculation, if necessary, use $1.

3. A business man, having heard much about correct depreciation, but understanding little of the methods of calculation used, calls on you to explain to him by means of comparison the five most important methods. As an illustration, use an asset costing $4000, with a scrap value of $400 and a life of 6 years, money being worth 6%.

4. A machine costing $81 is estimated to have a life of 4 years, with a residual value of $16. Prepare a statement showing the annual charge for depreciation according to each of the following methods:

1. Straight-line.
2. Constant-percentage-of-diminishing-value.
3. Annuity.

(For convenience in arithmetical calculation, assume the rate of interest to be 10%.)—*American Institute Examination.*

Depletion.—The provision for the extinction of wasting assets, such as mines, timber lands, or gravel pits, is called a provision for depletion.

There are two classes of problems with which we shall deal, namely, the determination of the amount of depletion each year and the capitalization of the wasting asset.

Calculation of depletion.—The amount of depletion usually stands in the same proportion to the total cost of the wasting asset as the units of product removed stand to the total units of product the asset was estimated to produce when new, but if the property is leased and it is not possible to remove all the product before the lease expires, it is plain that depletion should be based on the quantity to be extracted during the period of the lease.

Problems.

1. The mine superintendent of the Palo Alto Mining Company estimates the recoverable ore at 200,000 tons. The account, "Mining Property" shows a debit of $189,000.

What should be the depletion charge per ton on ore mined?

2. The Pana Coal Mining Company's "Coal Lands" are valued at $360,000. It is estimated that there are 12,000,000 recoverable tons of coal. During the year 1922, 95,060 tons were mined and surfaced. Compute the depletion charge for the year 1922.

Depletion of long leaf yellow pine timber, including naval stores rights.—As naval stores' rights are depleted or sold from a tract of pine timber at a value of $1 per 1000 feet, based on 50¢ per 1000 feet for the first year boxing, 30¢ per 1000 feet for second year boxing, and 20¢ per 1000 feet for third year boxing, the naval stores rights will be allocated out from the pine timber account and applied to such quantity as that on which the naval stores' rights are worked or leased. In this event, the depletable value of the timber, when cut, will be correspondingly reduced.

Problems.

1. The March 1, 1913, value of a tract of yellow pine timber, including turpentine rights, was:

428,177 thousand feet at $4.50..... $1,926,796.50.

Operations were begun in 1918, and during 1919 turpentine or naval stores' rights for first year boxing on 51,345 thousand feet of pine timber were sold.

In 1920, turpentine or naval stores' rights for first year boxing on 49,077 thousand feet, and for second year boxing on 51,345 thousand feet were sold.

In 1921, turpentine or naval stores' rights for second year boxing on 49,077 thousand feet, and for third year boxing on 51,345 thousand feet were sold.

In 1922, sales were made as follows: 49,077 thousand feet of third year boxing, and 4985 thousand feet first year boxing.

During the year 1922, 428 thousand feet at a cost of $4200 were added to the pine timber account.

The first timber sale was made in 1922, for 427,685 thousand feet. The pine timber account was closed in 1923, by the sale of 920 thousand feet.

Compute the amount of depletion for each year, beginning with 1919.

2. The cypress timber account at March 1, 1913, shows values as follows:

187,155,000 feet $962,563.75

SALES

1922.....	49,891,000 feet
1923.....	2,520,000 feet

Compute the depletion for each year.

Allocation of carrying charges.—The following problems show how the carrying charges are allocated:

Problems.

1. The Seminole Timber Company was formed for the purpose of purchasing, selling, and holding timber and timberlands.

The fair market value on March 1, 1913, for the land and timber owned was as follows:

Land, 72,041 acres at $1.00.....	$ 72,041.00
Timber:	
Pine, 428,177 M ft. at $4.50.....	1,926,796.50
Cypress, 187,155 M ft. at $5.25....	962,563.75
	$2,981,401.25

The company had no income during the years prior to 1918, and therefore capitalized the carrying charges as follows:

Nature of Expenditure	*Mar. 1, 1913, to Dec. 31, 1913*	*1914*	*1915*	*1916*	*1917*	*Total*
Taxes	$ 8,890.12	$ 9,291.86	$ 9,748.27	$10,786.73	$13,533.91	$ 52,250.89
Interest ...	8,650.98	8,914.79	8,462.17	7,687.50	8,870.65	42,586.09
Cruising ...	600.97	——	——	——	——	600.97
Title Exp..	——	521.90	2,023.07	2,053.16	1,995.21	6,593.34
Miscel.	60.00	270.21	228.94	147.80	149.92	856.87
Total	$18,202.07	$18,998.76	$20,462.45	$20,675.19	$24,549.69	$102,888.16

The total amount, $102,888.16, is allocated to the Land account, Pine account, and Cypress account, in accordance with the ratio that the value of each account bears to the total value, as of March 1, 1913.

Compute the ratios,* and make the allocation on a form like the following:

Year	*Total Carrying Charges*	Charges Allocated to *Land*	*Pine*	*Cypress*
1913				
1914				
1915				
1916				
1917				
Total				

* Carry Land and Pine ratios to the first decimal place, and the Cypress ratio to the second decimal. The three ratios should add to 100%.

2. Cost of sales in 1922 was as follows:

Cypress		$ 261,927.75
Yellow Pine	$1,824,162.05	
Turpentine *	12,307.90	$1,836,469.95

* See Depletion of Yellow Pine Timber, page 323.

What amount of carrying charges should be written off against 1922 income? Base your calculations on the facts given in problem 1.

Capitalization of wasting asset.—Occasionally it is necessary to find the capitalized value of a wasting asset. If the asset were a permanent asset, the value would be found as in a perpetuity, but as the asset has a limited life, it is necessary to provide a fund to replace it. This fund is usually in the form of a sinking fund.

To find the fair market value at which a wasting asset may be capitalized, divide the annual average income by the sum of, first, the result obtained by dividing 1 by the amount of an ordinary annuity at a fair rate of interest for the length of time expected to be required to deplete the mine, plus, second, a fair annual rate of income.

Example.—A certain mine produces annually an operating income of $10,000. At the present rate of depletion, it is estimated the mine will last for 8 years. If the sinking fund will earn 4%, and the owners are to receive a dividend rate of 6%, what should be the capitalized value?

Algebraic Formula.

$$\frac{\text{Operating profit}}{\dfrac{1}{\dfrac{(1+i)^n-1}{i}} + \text{Dividend rate}} = \text{Capitalized value.}$$

Arithmetical Substitution.

$$\frac{10000}{\dfrac{1}{\dfrac{(1.04)^8-1}{.04}} + .06} = \text{Capitalized value.}$$

Solution.

$(1.04)^8 = 1.3685691$, compound amount of 1 for 8 periods at 4%.
$1.3685691 - 1 = .3685691$, compound interest on 1 for 8 periods at 4%.
$.3685691 \div 4 = 9.214226$, amount of annuity of 1 for 8 periods at 4%.
$1. \div 9.214226 = .1085278$, the rent of an annuity that will amount to 1.
$.1085278 + .06 = .1685278$, rent of annuity plus the dividend rate.
$10000 \div .1685278 = 59{,}337.39$, capitalized value.

TABLE

	Annual Income	*6% on Capitalized Value*	*Sinking Fund Payments*	*Interest on Sinking Fund*	*Sinking Fund Accumulations*
1	$10,000	$3,560.24	$6,439.76		$ 6,439.76
2	10,000	3,560.24	6,439.76	$ 257.99	13,137.11
3	10,000	3,560.24	6,439.76	525.48	20,102.35
4	10,000	3,560.24	6,439.76	804.09	27,346.20
5	10,000	3,560.24	6,439.76	1,093.85	34,879.81
6	10,000	3,560.24	6,439.76	1,395.19	42,714.76
7	10,000	3,560.24	6,439.76	1,708.59	50,863.11
8	10,000	3,560.24	6,439.76	2,034.52	59,337.39

Problems.

1. Find the capitalized value of a coal mine which will produce a net income of $25,000 a year for thirty years, if the annual income rate is 6%, and a sinking fund is to be accumulated at 4%.

2. A tract of timber will yield an annual revenue of $20,000 for 25 years. If annual dividends are declared at 5%, and payments are made annually into a sinking fund which bears 4% interest, what is the value of the timber rights?

3. A gravel pit is estimated to contain 3,000,000 cubic yards of gravel. This pit is leased at a royalty of $.20 per cubic yard extracted. If the average annual output is 150,000 cubic yards, what should be the capitalized value in order to pay a 6% dividend on the stock, and provide for a fund, at 4%, equal to the capital stock?

Chapter 32

BUILDING AND LOAN ASSOCIATIONS

Organization.—Building and loan associations are organized under the laws of the State in which they are located. In many States, their affairs are examined periodically by the State Bank Examiner in the same manner as are all savings banks and State banks. A copy of the State law may be obtained by writing to the Superintendent of Banks of your State.

Membership.—Anyone may become a member of a building and loan association by agreeing to save or invest a certain amount each week or month (usually monthly). Each member, on opening an account, receives a pass book in which are credited his deposits each month.

Fully-paid shares.—Fully-paid shares may be issued for one payment of $100 each (the par value of a share is usually $100, although other amounts are possible). A "fully-paid certificate" is issued for the number of shares bought, and dividends are paid in cash each six months.

Withdrawal of funds.—In most cases, funds deposited in an association must be left for at least six months. After that time any member may withdraw all or a part of his funds, together with the dividends credited and not already paid. In event of withdrawal before the maturity of the shares, some associations retain a membership or withdrawal fee of 2% of the par value of each share for which they issue membership certificates which are transferable upon the books of the association. The member may retain this certificate, or assign it to anyone as he sees fit. When another account is opened in the association, either by himself or anyone else holding the certificate of membership, credit will be given for the amount it represents.

Other associations permit the withdrawal of the amount of the deposits and allow the holder interest for the equated

time. The rate of interest is fixed by the association and is usually lower than the per cent earned by the shares. This difference in rates results in a profit to the association. Such profit on withdrawals is added to the other profits and distributed to the shares remaining in the association.

Form of association.—The life of the association is limited under the terminating plan to any number of years that may be agreed upon. When the specified number of years elapses, the association goes out of existence or a new one may be formed for another period of time. The monthly payment necessary to retire the stock is determined in the same manner as is the rent of the present worth of an annuity.

Determination of the amount to be paid monthly under the terminating plan.—

Procedure.—(*a*) Compute the interest rate per period by dividing the annual rate by the number of times converted.

(*b*) Compute the number of periods by multiplying the time in years by the number of times the interest is converted annually.

(*c*) Compute the present value of an annuity of 1, using the periodic rate found in (*a*) and the number of periods as found in (*b*).

(*d*) Divide the par value of a share by the present value of an annuity of 1 found in (*c*).

Example.—What should be the monthly payment per share, each share having a par value of $100, if it is desired to mature the shares in 5 years, money being worth 6%?

Algebraic Formula.

$$\frac{\dfrac{P}{1-\dfrac{1}{(1+i)^n}}}{i} = \text{Monthly payment.}$$

Arithmetical Substitution.

$$\frac{\dfrac{\$100}{1-\dfrac{1}{(1.005)^{60}}}}{.005} = \text{Monthly payment.}$$

Solution.

$.06 \div 12$	$= .005,$	monthly rate of interest.
5×12	$= 60,$	number of periods.
$(1.005)^{60}$	$= \$1.3488502,$	compound amount of 1 at .005 for 60 periods.
$1 \div 1.3488502$	$= .741372,$	present worth of 1 at .005 for 60 periods.
$1 - .741372$	$= .258628,$	compound discount on 1 at .005 for 60 periods.
$.258628 \div .005$	$= 51.7256,$	present worth of annuity at .005 for 60 periods.
$\$100 \div 51.7256$	$= \$1.933+,$	periodic rent.
$\therefore \$1.93+$	= the monthly payment.	

Under this plan the association would be dissolved at the end of the 5-year period.

Serial plan.—Under this plan stock is issued at specified dates. Each issue constitutes a new series and shares in profits in proportion to the length of time the series is outstanding. Distribution of profits under this plan is similar to the distribution of profits in a partnership using the average investment method. Since average investment is illustrated in the chapter on partnerships, it will not be repeated here.

A new series is issued semiannually or annually according to the demand for the stock, and as soon as a new series is opened, issuance of shares in the previous series is closed.

Partnership plan.—The name indicates the substance of this plan, and an example and solution are given to show its application.

Example.—The Twin City Building and Loan Association has issued six series of shares, as follows:

Series	*Date*	*Number of Shares*
1	Jan. 1, 1920	500
2	July 1, 1920	500
3	Jan. 1, 1921	400
4	July 1, 1921	300
5	Jan. 1, 1922	400
6	July 1, 1922	400

The dues in each series were $1 a share a month. The net profit from interest, fines, and so forth, less the operating expenses for the half year ending December 31, 1922, was $965.22, and the undivided profits on July 1, 1922, were $4272.78.

The status of the shares July 1, 1922, was as follows:

Series	*Date of Issue*	*Number of Shares*	*Paid per Share*	*Profit per Share*	*Value per Share*
1	Jan. 1, 1920	500	$30	$4.12	$34.12
2	July 1, 1920	500	24	2.66	26.66
3	Jan. 1, 1921	400	18	1.51	19.51
4	July 1, 1921	300	12	.69	12.69
5	Jan. 1, 1922	400	6	.19	6.19

Distribute the profits for the half year ending December 31, 1922, by the partnership plan.

Solution.

On December 31, 1922, the amount paid on each share of each series was as follows:

Series	*Paid per Share*
1	$36
2	30
3	24
4	18
5	12
6	6

Dues of $1 at the beginning of the month have been paid on each of the 500 shares in the first series for 36 months. The average time is found to be 18½ months, that is, $1 was invested for 36 months; $1 for 35 months; $1 for 34 months, and so forth, the average, 18½ months, being the sum of the first term plus the last term divided by 2, $\frac{36+1}{2}$ = 18½ months. Solving for series 2, 3, 4, 5, and 6, we have 15½, 12½, 9½, 6½, and 3½ respectively, as the average time for each series.

As there are 500 shares in the first series, and each share has paid in $36, the capital of the first series is $18,000. $18,000 for 18½ months equals $333,000 for one month. Solving for each series, we have the following solution:

Series						*Dollars for 1 Month*
1	$36 × 18½	months	× 500	shares	=	$333,000
2	30 × 15½	"	× 500	"	=	232,500
3	24 × 12½	"	× 400	"	=	120,000
4	18 × 9½	"	× 300	"	=	51,300
5	12 × 6½	"	× 400	"	=	31,200
6	6 × 3½	"	× 400	"	=	8,400
						$776,400

Undivided profits July 1, 1922	=	$ 4,272.78
Earnings July 1, 1922-December 31, 1922	=	965.22
Profits available for distribution	=	$ 5,238.00

The profits available for distribution, $5238, are prorated among the series in the proportion that each one's equated capital bears to the total equated capital as follows:

Series											
1	$\frac{3330}{7764}$	of	$5238 =	$2246.59	for	500	shares,	or	$4.49	per	share.
2	$\frac{2325}{7764}$	"	5238 =	1568.57	"	500	"	"	3.14	"	"
3	$\frac{1200}{7764}$	"	5238 =	809.58	"	400	"	"	2.02	"	"
4	$\frac{513}{7764}$	"	5238 =	346.10	"	300	"	"	1.15	"	"
5	$\frac{312}{7764}$	"	5238 =	210.49	"	400	"	"	.53	"	"
6	$\frac{84}{7764}$	"	5238 =	56.67	"	400	"	"	.14	"	"
				$5238.00							

Dexter's rule for distribution of profits.—This plan is a modification of the partnership plan, and its principles are shown in the following solution based on the problem given under the partnership plan.

Solution.

To the capital of each series on July 1, 1922, as shown by the column "value per share" in the previous tabulation, should be added the contribution of $1 per share for each of the six months of the current half year computed by the average method.

$1	paid	July	1 =	$1	for	6	months,	or	$6	for	1 month.
1	"	Aug.	1 =	1	"	5	"	"	5	"	1 "
1	"	Sep.	1 =	1	"	4	"	"	4	"	1 "
1	"	Oct.	1 =	1	"	3	"	"	3	"	1 "
1	"	Nov.	1 =	1	"	2	"	"	2	"	1 "
1	"	Dec.	1 =	1	"	1	"	"	1	"	1 "
									$21	"	1 "

$21 ÷ 6 = $3.50, average for 6 months.

Add $3.50 to the value of each share, and multiply by the number of shares in the series to find the capital value of the series.

Series	*Shares*	*Dollars per Share*	*Capital per Series*
1	500	$37.62	$18,810.00
2	500	30.16	15,080.00
3	400	23.01	9,204.00
4	300	16.19	4,857.00
5	400	9.69	3,876.00
6	400	3.50	1,400.00
		Total Capital	$53,227.00

Previous distributions of profits are unchanged. The profits of the current period are added to them on the basis of the present earning capitals; in other words, the capital \$53,227 earned a profit of \$965.22, or 1.813403%. Hence the profit for each series and each share in a series is calculated as follows:

Series	*Capital*	*Rate*	*Profit per Series*	*Shares*	*Profit per Share for Last Period*
1	\$18,810	.018134	\$341.10	500	\$.68
2	15,080	.018134	273.46	500	.55
3	9,204	.018134	166.91	400	.42
4	4,857	.018134	88.08	300	.29
5	3,876	.018134	70.29	400	.18
6	1,400	.018134	25.38	400	.06
			\$965.22		

Adding the profit per share for the last period to the value of the share as of July 1, 1922, and the monthly payments for the period gives the value per share as of December 31.

Comparison of partnership plan and Dexter's rule.—

	Partnership Plan			Dexter's Rule		
Series	*Profit per Share, July 1, 1922*	*Profit per Share, Dec. 31, 1922*	*Profit per Share for Last 6 Mos.*	*Profit per Share for Last 6 Mos.*	*Difference Inc.*	*Dec.*
1	\$4.12	\$4.49	\$.37	\$.68	\$.31	
2	2.66	3.14	.48	.55	.07	
3	1.51	2.02	.51	.42		\$.11
4	.69	1.15	.46	.29		.17
5	.19	.53	.34	.18		.16
6	.00	.14	.14	.06		.08

The above comparison shows that the partnership plan favors the newer series at the expense of the old. This is unjust because it was really the old series that produced the profits; therefore Dexter's Rule is favored for the distribution of profits in a building and loan association using the serial plan.

Problems.

Partnership Plan.

1. A building and loan association issued a new series each year, as follows:

Series	*Date of Issue*	*Number of Shares*
1	Jan. 1, 1918	300
2	Jan. 1, 1919	400
3	Jan. 1, 1920	500
4	Jan. 1, 1921	300

The dues were \$1 per month per share. The profits to the end of the fourth year were \$4200. Find the value of one share in each series at the end of the fourth year.

2. The first series of a certain building and loan association is 3 years old and has 1000 shares; the second series is 2 years old and has 500 shares; and the third series is 1 year old and has 400 shares. The net assets are \$59,860. Payments were \$1 a share a month.

Find:

(*a*) Net profits.
(*b*) Profit for 1 share in each series.
(*c*) Value of 1 share in each series.

3.

Dexter's Rule.

STATEMENT, DECEMBER 31, 1920

Series	*Date Issued*	*Number of Shares*	*Paid per Share*	*Profit per Share*	*Value per Share*
1	Jan. 1, 1919	500	\$24	\$3.32	\$27.32
2	July 1, 1919	400	18	1.89	19.89
3	Jan. 1, 1920	300	12	.86	12.86
4	July 1, 1920	400	6	.23	6.23

The fifth series was issued January 1, 1921, for 300 shares. The dues in each series were \$1 a share a month. The profits for the six months ending June 30, 1921, after deducting all expenses, amounted to \$562.75. Find the value of 1 share in each series as of June 30, 1921.

4. The sixth series was issued July 1, 1921, for 400 shares. The net profits for the six months ending December 31, 1921, were \$604.85. Find the value of 1 share in each series, as of December 31, 1921.

Withdrawal value.—If 20 shares of series 3 in the problem on page 330 are withdrawn October 1, 1922, what is their withdrawal value, assuming the association allows 5% interest on shares withdrawn before maturity?

Each share of the third series has a paid up value on October 1, 1922, of $21; hence the 20 shares are worth $420. To this amount add 5% interest for the equated time, 11 months, $\frac{21+1}{2}$, or $19.25, giving a withdrawal value of $439.25. The book value of each share of this series is $22.51 ($21 in each payment plus $1.51 profits as of July 1, 1922). Twenty shares at $22.51 gives a book value of $450.20. The difference, $10.95, is a profit on the transaction and makes the divisible profits for the half year $976.17 ($965.22 + $10.95).

Problems.

Based on illustrative problem, page 330.

1. Find the withdrawal value of 10 shares of series 2, withdrawn December 1, 1922, interest allowed at 5%.

2. Find the withdrawal value of 20 shares of series 4, withdrawn November 1, 1922, interest allowed at 4%.

3. Find the withdrawal value of 10 shares of series 3, withdrawn July 1, 1922, assuming that the association permits the withdrawal of the book value, but retains a membership fee of 2% of the par value of each share.

Dayton or Ohio plan.—The most popular plan in use in building and loan associations is the Dayton or Ohio plan. Use of this plan removes the uncertainty of the time of maturity of a loan, thus giving the borrower a definite contract. Under other plans a successful association would soon mature its stock, while an unsuccessful one would greatly prolong the time during which the borrower would have to continue his interest payments; furthermore, under the Dayton plan the borrower does not have to own stock. Each payment that the borrower makes does two things: first, it pays the interest; second, it reduces the principal. This principle is applied in the Federal Farm Loan Act and is an application of the subject "Installment Payments" previously discussed. The monthly payment is found by the formula used under the terminating plan.

Problems.

1. Find the monthly payment which will cover both principal and interest on a $3000 loan at 6% for 10 years.

2. Construct a schedule for the first two years.

3. Construct a schedule for the last two years, showing the final amortization of the debt.

Classified Problems.

Distribution of profits to shareholders.—

1. Five shares of stock in a building and loan association have a book value of $189.60 at the beginning of a 6 months' period. The dues of $5 per month in advance for the next 6 months were paid when due. What is the average book value of the period that should be used in the distribution of profits to these 5 shares?

2. I subscribed for 30 shares in a building and loan association between dividend rates so that I had to pay my $30 in dues each month for 4 months. What is the book value of my payments?

3. Twenty shares of stock had a book value of $500 at the beginning of a 6 months' period. The shareholder became delinquent for three months. At the beginning of the fourth month he paid $80, and $20 each month for the next 2 months. What was the average book value of his 20 shares? If his delinquency had been covered by fines so that he could have had full participation in profits, what would have been the book value?

4. Ten shares of stock in *X* Building and Loan Association have a book value of $291.71 on July 1, 1921. Dues of $1 a month a share were paid for the next 6 months. December 31, 1921, the average book value of holdings in the association was $126,178.36. The association reported a net gain for the 6 months amounting to $4037.71. Find (*a*) the rate per cent earned; (*b*) the dividend on the 10 shares, December 31, 1921; and (*c*) the book value of the 10 shares, December 31, 1921.

5. Adams paid $65 in advance for 1 share of paid-up stock in the Ames Building and Loan Association, the maturity value of the shares being placed at $100. The per cent of earnings for the 5 succeeding semiannual periods was 4.6%, 3.9%, 4.2%, 5.1%, and 4.9%. What was the book value of Adams' share at the beginning of the sixth period?

6. The Garford Building and Loan Association issued shares at the beginning of each month. Smith subscribed for 30 shares just 1 month before the end of a 6 months' period. He paid for these shares at the rate of $30 a month. What is the average book value of these 30 shares to be used in the distribution of profits for the 6 months' period? How much was Smith entitled to receive as dividends if the Association showed a net gain of $2631.25, and a book value of $85,160.72?

Shares issued in series.—

1. *B* has 10 shares of $100 each in each of 2 series of the Capital Building and Loan Association. The 10 shares are of a series which is ending its second 6 months' period, and 10 shares are of a series which is ending its first 6 months' period. *B* has paid his dues of $10 a month on each group of shares. The Association's rate of profit for the 6 months' period just ended is 4½%. What are the dividends on each group of shares?

Withdrawal values.—

1. Lee has paid $20 a month on 20 shares of stock in the Midway Building and Loan Association for 54 payments. When the 55th payment was due, he withdrew his money for the withdrawal value. The Association allowed him the sum of his payments and simple interest at 6% per annum. The value of the stock had been accumulating at 7%, convertible monthly. What was the book value of the stock? What was the withdrawal value? How much profit was retained by the Association?

2. If the interest paid on withdrawals had been calculated at 5%, what would have been the difference between book value and withdrawal value?

3. If the stock had been accumulating at 6%, convertible monthly, and simple interest at 6% was paid on withdrawals, what would have been the difference between book value and withdrawal value?

4. A stockholder, making payments at the rate of $10 a month for 85 months, withdraws at the date of the 86th payment. The stock has been accumulating at the rate of 6½%. The Association allows 4% simple interest on the payments withdrawn. What is the difference between the book value and the withdrawal value if the stock has a maturity value of $200 a share?

Time required for stock to mature (rate of interest given).—Such a problem is simply that of finding the time for an annuity certain of annual rents in twelve monthly payments, to accumulate to a certain amount.

Procedure.—(*a*) Divide the maturity value by the number of dollars in the periodic payment.

(*b*) Multiply the quotient found in (*a*) by the periodic rate per cent.

(*c*) Determine the logarithm of 1 plus the product found in (*b*).

(*d*) Determine the logarithm of 1 plus the periodic rate per cent.

(*e*) Divide the logarithm of the number found in (*c*) by the logarithm of the number found in (*d*).

Example.—The Federal Building and Loan Association yields the investor a nominal rate of 7%, convertible monthly. What is the time required for payments of $1 a month to mature $100?

Algebraic Formula.

$$\frac{\text{Log}\left[1+\left(\frac{\text{Maturity value}}{\text{Periodic payment}}\times i\right)\right]}{\text{Log}\quad 1+i}=\text{Term.}$$

Arithmetical Substitution.

$$\frac{\text{Log}\left[1+\left(\frac{100}{1}\times .0058333\right)\right]}{\text{Log}\quad 1.0058333}=\text{Term.}$$

Solution.

Dividing,	$100 \div 1$	$= 100.$	(1)
Multiplying,	$100 \times .0058333$	$= .58333.$	(2)
Adding,	$1 + .58333$	$= 1.58333.$	(3)
Log,	1.58333	$= 0.199572.$	(4)
Log,	1.0058333	$= 0.002526 +.$	(5)
Dividing,	Log $0.199572 \div$ Log $0.002526 +$	$= 79$ (approximately).	(6)

Problems.

1. If the nominal rate of interest in the above example had been 6%, what would be the time of maturity?

2. If the monthly payment is 50 cents and the nominal rate 6%, convertible monthly, what time is required to mature a $100 share?

3. If the payments are 50 cents a month on a $100 share and earn for the investor the effective rate of 7%, convertible annually, in what time will the stock mature?

The effective rate of interest on money invested in installment shares.—This is a problem illustrating the rate earned on an annuity.

Example.—The Central Building and Loan Association issued $100 par value stock with monthly payments of $1 per share, July 1, 1915. The stock matured January 1, 1922, by the association accepting two cents on the 80th payment of $1. What was the effective rate of interest earned?

To find the effective rate it is necessary to use estimated rates as explained in "Finding the rate of the amount of an ordinary annuity." The formula for the first estimated rate is:

Algebraic Formula.

$$P\left(\frac{(1+i)^n - 1}{i}\right) = \text{Amount.}$$

Arithmetical Substitution.

$$1\left(\frac{(1.005666)^{79} - 1}{.005666}\right) = \text{Amount.}$$

Solution

First trial rate, using 6.8% as trial:

$6.8 \div 12$	$=$	$.005\frac{2}{3}$,	monthly rate.
$1.005\frac{2}{3}$ to 79 power (by logs)	$=$	1.5625808,	compound amount.
$1.5625808 - 1$	$=$	.5625808,	compound interest.
$.5625808 \div .005\frac{2}{3}$	$=$	$99.28,	amount of annuity.

Trial rate is found to be too small.

Second trial rate, using 7.2% as trial:

$7.2 \div 12$	=	.006,	rate per month.
1.006 to 79 power (by logs)	=	1.6041402,	compound amount.
$1.6041402 - 1$	=	.6041402,	compound interest.
$.6041402 \div .006$	=	$100.69,	amount of annuity.

Trial rate is found to be too large.

INTERPOLATION

Value at 7.2%	=	$100.69
Value at 6.8%	=	99.28
Difference represented by .4%	=	1.41

$1.41 \div .4 = .35$, the difference represented by .1%.

$100 — .02, excess of last payment	=	$ 99.98
Value of 79 payments at ? rate	=	99.98
Value of 79 payments at 6.8%	=	99.28
Excess over rate of 6.8%	=	.70
Difference in value at .1%	=	.35

$.70 \div .35 = 2$, or .2 to be added to 6.8%, or 7%.

Problems.

1. If in the above example the stock had matured by an acceptance of the full amount of the 75th payment, what would have been the effective rate earned?

2. Payments of 50 cents a month mature $100 in 10 years, 4 months, without accepting any part of the 125th payment. What is the effective rate of interest earned?

The interest rate from the borrower's standpoint.—

1. A certain building and loan association, operating on a 7% nominal interest basis, will finance the building of your house. By making payments of $1 a month on a $100 share, your loan will mature on the 79th monthly payment. The Equitable Life Insurance Company, through its financial agents in this city is offering loans on real estate at 5½%. If you should pay the 5½% interest monthly in advance, and invest the difference between this amount and the $1 share payable to the building and loan association in a sinking fund at 4% interest, payable monthly, which would prove to be the better proposition at the end of 78 months?

2. If the difference in the monthly payments were placed in a savings bank at 4%, convertible semiannually, which would prove to be the better proposition?

The interest rate from the borrower's standpoint when the interest and dues are together considered as a single sum for the payment of interest and principal.—

1. Jones borrowed $1000 from an association operating on the basis of a 6% nominal interest rate convertible monthly. Each month he paid $5 dues and $5 interest. His stock matured at the end of 11 years and 3 months, after making the monthly payment of dues and interest required at that time. What effective rate of interest has he paid on his loan?

Note.—It will be necessary to use an estimated rate and solve by interpolation.

2. An association is operating on a 7% basis. Smith borrows $1200, paying $7 interest and $12 dues each month. The 79th payment was $7 interest and $2.40 dues. This was the final payment and matured the stock. What was the effective interest rate?

APPENDICES

Appendix A—Miscellaneous Problems

1. Compute the simple interest for 8 years, 6 months on $3,467.50, at 4.684%. (Compute by the use of logarithms.)

2. Simplify the following expression:

$$\$50\left(\frac{1-\frac{1}{(1.005)^{12n}}}{.005}\right)=\$4140.$$

3. Compute the compound amount of $465 for 17 years at 4%, interest compounded annually.

4. What sum must be placed in an endowment fund, which is invested in 20-year, 5½% bonds, to produce an annual rent of $600 for a period of 20 years?

5. *B* buys a house, agreeing to pay $600 cash and $600 the first of each year for 9 years thereafter. If money is worth 6% annually, what is the present value of the house?

6. John Smith buys a farm, agreeing to pay $10,000 for it in 10 equal annual installments, the first payment to be made immediately. Money is worth 5% annually. What is the annual payment?

7. A $1000 bond is purchased at $1041. This bond has 10 years to run and pays 6% interest annually. What is the rate of interest on the purchaser's investment?

8. Find the rate of discount, when the cash rate of interest is 8%, convertible quarterly.

9. The principal was $500, the amount $600, and the time 15 years. The interest was compounded semiannually. What was the rate?

10. A 10-year bond of $1000 face value, bearing 5% interest payable semiannually, is purchased to net 6%. What is the purchase price?

11. If interest is converted quarterly at 5%, what is the annual effective rate?

12. Without the use of a compound interest table, show two short methods of computation for finding the compound amount of $1 at 6% for 32 years.

13. Give a formula for computing the present worth of an ordinary annuity for 12 years at 6%.

14. What is the capitalized value of a farm, if it produces $1000 after paying for upkeep? Money is worth 6% per annum.

15. Give a formula for the amount of an ordinary annuity of 1 for 8 years at 6%.

16. Construct a table showing carrying value, depreciation charge, depreciation reserve, and the net value of the asset for a machine costing $2000, scrap value $300, life 15 years. Use the fixed-percentage-of-diminishing-value method.

17. If money is worth 5% per annum, construct a columnar table for the annuity method of depreciation, using the information given in problem 16 above.

18. Argument has been strongly urged that, aside from any question of possible mismanagement, or of the difficulty of making satisfactory investments which yield the same rate as is paid on the bonds, a sinking fund for bonds is more expensive than an arrangement for the serial repayment of bonds. This is illustrated by the case of $20,000, 5% bonds. If these are paid off in a series, one each year, the total payment made will be, principal $20,000, interest $10,500, total $30,500. The annual sinking fund to pay these bonds would be on a 5% basis amounting to $604.85, making in 20 years $12,097, and the interest paid on the bonds would be $20,000, total payments $32,097. The apparent excess burden is accordingly $1597.

Discuss the above argument and show clearly just what the figures mean, and in what the apparent saving actually consists.—*American Institute Examination.*

19. The Ramsey County Savings Bank purchased on an interest date $100,000 par value of L. & O. first mortgage

bonds. The bonds have 3 years to run, and bear interest at the rate of 5% per annum, payable semiannually, and were purchased at such a price as to yield net on the investment 4% per annum, payable semiannually. Find the cost of the bonds and construct a table showing their book values on successive interest dates to maturity, the effective interest, and the amortization of the premium; also indicate the entries that should be made at the collection of the first interest coupon.

The following tables are given to assist in the computations:

Periods	*Amount of $1 at 2% Compound Interest*	*Present worth of $1 at 2% Compound Interest*
0	1.	1.
1	1.02	0.08039216
2	1.0404	0.96116878
3	1.061208	0.94232233
4	1.08243216	0.92384543
5	1.10408080	0.90573081
6	1.12616242	0.88797138
7	1.14868567	0.87056018

C.P.A.,—Minnesota.

20. If a company hires a man at $50 a month to guard a crossing, how much could it afford to pay to construct an overhead crossing, if money is worth 6% annually?

21. Find the value of $(1.04)^{80}$.

22. Find the amount of $200 with interest at 4%, compounded quarterly, for 60 years.

23. Mr. Davis desires to know how much money must be placed in the bank the day his son is 5 years old in order that on the day his son is 21 the fund will have accumulated to $10,000. The bank pays 4% interest, compounded quarterly.

24. Construct a columnar table showing the increase of a fund, the total value, and amount of interest each year for 8 years, if the annual contribution is $100, and money is worth 6% compounded semiannually.

25. *A* wishes to retire a debt of $8000, bearing 8% interest, payable annually, by making semiannual payments of

equal amounts at the end of each 6 months for 8 years. What should be the amount of each payment?

26. What would be the annual payment in the above problem, if the payments were made at the beginning of each year?

27. A company desired to know how much must be paid into a fund at the end of each year to retire a bond issue of $20,000, due in 15 years. It is assumed that the fund will earn 4% for the first 5 years, 5% the second 5 years, and 6% the last 5 years.

28. How much will have accumulated at the end of 5 years, from payments of $300 made annually, if the bank allows 4% compounded quarterly?

29. *A* and *B* are dealers in bonds and share profits in the following proportion: *A*, 75%; *B*, 25%. *A* and *B* engage *C* to sell bonds, agreeing to pay him a salary equal to 25% of the net profits to be divided between the partners.

During the continuance of *C*'s contract, the firm purchases one hundred thousand dollars ($100,000) Waterville Traction Company, first mortgage 5% bonds on a 3% basis. The bonds have eighteen (18) months to run, interest payable semiannually (three interest periods).

The firm holds the Waterville bonds till maturity. Prepare a statement of the Waterville Bond accounts, showing cost, interest, and amortization.—*C. P. A.—New York.*

30. If it is necessary to rebuild a certain bridge every 25 years and it costs $50,000 each time, what sum of money must be set aside annually to provide for rebuilding the bridge, if money is worth 4% annually?

31. A music house sells pianos on the installment plan, receiving monthly payments. Pianos cost $600 and are sold for $900. Interest at 6% a year, or ½% a month, is charged on unpaid balances of the purchase price. Income each year is computed by taking such a proportion of the total profit as the collections on principal during the year bear to the $900 selling price. In addition, income is credited with interest collected and accrued during the year.

A piano is sold August 16, 1921. An initial payment of $50 is received and the purchaser agrees to pay the balance

in 30 equal monthly installments, payable on the 16th of each month, beginning September 16, 1921. These equal monthly payments are to be applied first to the interest accrued during the month and second to the principal. As the interest will decrease each month, the payment on principal will increase.

(*a*) Compute the monthly payment to be made. (The present value of $1 due 30 periods hence at ½% per period is $.8610297.)

(*b*) Set up a table with the following columns:

Date	*Payments*			*Balance of Principal*
	Total	*Interest*	*Principal*	

Enter the payments received in August, September, October, November and December and show the balance unpaid at December 16, 1921.

(*c*) State the total credit to income during 1921, including profit on sale and interest.—*American Institute Examination.*

32. The combined annual gross and net earnings of the operated public utility companies of the Standard Gas and Electric Company have been as follows:

For 12 Months Ended December 31	*Gross*	*Net*
1914	$16,435,288	$ 7,780,367
1915	17,723,057	8,729,888
1916	19,677,755	9,601,286
1917	22,152,152	10,000,292
1918	26,250,856	10,078,818
1919	30,605,747	11,622,074
1920	36,246,956	12,694,504
1921	39,321,196	14,251,274
1922	42,038,742	16,056,229

(*a*) What have been the average gross earnings per year?

(*b*) What have been the average net earnings per year?

33. A man bought a house on contract for $3500. He agreed to make a payment of $75 each month to pay the interest on the principal to date, and the balance was to apply on the retirement of the debt. He was to receive a deed for the property when the principal was paid. After he had made his regular monthly payments for two years and nine months, he decided to sell his place and desired to know how much he owed. How much did he owe?

STEPHEN F. AUSTIN STATE TEACHERS COLLEGE
NACOGDOCHES, TEXAS

TABLE OF LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9	D
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
105	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
115	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
135	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702	314
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911	309
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982	307
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068	301
145	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689	289
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558	287
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051	281
155	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	310056	310268	310481	310693	310906	311118	311330	311542	212
205	311754	311966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012	207
210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	330008	330211	203
4	330414	330617	330819	331022	331225	331427	331630	331832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246	199
9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	344196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	350054	194
4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835	360025	360215	360404	360593	360783	360972	361161	361350	361539	189
230	361728	361917	362105	362294	362482	362671	362859	363048	363236	363424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
235	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030	181
240	380211	380392	380573	380754	380934	381115	381296	381476	381656	381837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	390051	390228	390405	390582	390759	177
6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	398114	398287	398461	398634	398808	398981	399154	399328	399501	173
1	9674	9847	400020	400192	400365	400538	400711	400883	401056	401228	173
2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	410102	410271	410440	410609	410777	410964	411114	411283	411451	169
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	415140	415307	415474	415641	415808	415974	416141	416308	416474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	420121	420286	420451	420616	420781	420945	421110	421275	421439	165
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
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7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
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6	440909	441066	441224	441381	441538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
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6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
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6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
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6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
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6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
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6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
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1	9374	9491	9608	9725	9842	9959	570076	570193	570309	570426	117
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3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
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6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
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6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326	102
7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
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8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245	91
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151	91
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4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107	89
490	690196	690285	690373	690462	690550	690639	690728	690816	690905	690993	89
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
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4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
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6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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2	700704	700790	0877	0963	1050	1136	1222	1309	1395	1482	86
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
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6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
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1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	710033	85
3	710117	710202	710287	710371	710456	710540	710625	710710	710794	0879	85
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
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2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
525	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	724276	724358	724440	724522	724604	724685	724767	724849	724931	725013	82
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
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6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
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6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
550	740363	740442	740521	740600	740678	740757	740836	740915	740994	741073	79
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
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6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
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4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
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6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
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2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970	71
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
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3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
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6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
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2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	819544	819610	819676	819741	819807	819873	819939	820004	820070	820136	66
1	820201	820267	820333	820399	820464	820530	820595	0661	0727	0792	66
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
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7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	838849	838912	838975	839038	839101	839164	839227	839289	839352	839415	63
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	840043	63
2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	63
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
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7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
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3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
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2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930	58
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
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2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
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7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
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9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
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4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
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6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
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9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
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8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
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3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
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6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
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3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
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6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
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7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549	52
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
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4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
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6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
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2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
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6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879	51
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389	51
2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898	51
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
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6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
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9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
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8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
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4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
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7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829	49
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2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
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1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
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4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
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6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
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1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212	47
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2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903	47
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300	47
4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543	46
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

$(1 + i)^n$

n	⅛%	¼%	⅜%	½%	⅝%
1	1.0012 5000	1.0025 0000	1.0037 5000	1.0050 0000	1.0062 5000
2	1.0025 0156	1.0050 0625	1.0075 1406	1.0100 2500	1.0125 3906
3	1.0037 5469	1.0075 1877	1.0112 9224	1.0150 7513	1.0188 6743
4	1.0050 0938	1.0100 3756	1.0150 8459	1.0201 5050	1.0252 3535
5	1.0062 6564	1.0125 6266	1.0188 9115	1.0252 5125	1.0316 4307
6	1.0075 2348	1.0150 9406	1.0227 1200	1.0303 7751	1.0380 9081
7	1.0087 8288	1.0176 3180	1.0265 4717	1.0355 2940	1.0445 7891
8	1.0100 4386	1.0201 7588	1.0303 9672	1.0407 0704	1.0511 0753
9	1.0113 0641	1.0227 2632	1.0342 6070	1.0459 1058	1.0576 7695
10	1.0125 7055	1.0252 8313	1.0381 3918	1.0511 4013	1.0642 8743
11	1.0138 3626	1.0278 4634	1.0420 3220	1.0563 9583	1.0709 3923
12	1.0151 0356	1.0304 1596	1.0459 3983	1.0616 7781	1.0776 3260
13	1.0163 7244	1.0329 9200	1.0498 6210	1.0669 8620	1.0843 6780
14	1.0176 4290	1.0355 7448	1.0537 9908	1.0723 2113	1.0911 4510
15	1.0189 1495	1.0381 6341	1.0577 5083	1.0776 8274	1.0979 6476
16	1.0201 8860	1.0407 5882	1.0617 1739	1.0830 7115	1.1048 2704
17	1.0214 6383	1.0433 6072	1.0656 9883	1.0884 8651	1.1117 3221
18	1.0227 4066	1.0459 6912	1.0696 9521	1.0939 2894	1.1186 8053
19	1.0240 1909	1.0485 8404	1.0737 0656	1.0993 9858	1.1256 7229
20	1.0252 9911	1.0512 0550	1.0777 3296	1.1048 9558	1.1327 0774
21	1.0265 8074	1.0538 3352	1.0817 7446	1.1104 2006	1.1397 8716
22	1.0278 6396	1.0564 6810	1.0858 3111	1.1159 7216	1.1469 1083
23	1.0291 4879	1.0591 0927	1.0899 0298	1.1215 5202	1.1540 7902
24	1.0304 3523	1.0617 5704	1.0939 9012	1.1271 5978	1.1612 9202
25	1.0317 2327	1.0644 1144	1.0980 9258	1.1327 9558	1.1685 5009
26	1.0330 1293	1.0670 7247	1.1022 1043	1.1384 5955	1.1758 5353
27	1.0343 0419	1.0697 4015	1.1063 4372	1.1441 5185	1.1832 0262
28	1.0355 9707	1.0724 1450	1.1104 9251	1.1498 7261	1.1905 9763
29	1.0368 9157	1.0750 9553	1.1146 5685	1.1556 2197	1.1980 3887
30	1.0381 8768	1.0777 8327	1.1188 3682	1.1614 0008	1.2055 2661
31	1.0394 8542	1.0804 7773	1.1230 3245	1.1672 0708	1.2130 6115
32	1.0407 8478	1.0831 7892	1.1272 4383	1.1730 4312	1.2206 4278
33	1.0420 8576	1.0858 8687	1.1314 7099	1.1789 0833	1.2282 7180
34	1.0433 8836	1.0886 0159	1.1357 1401	1.1848 0288	1.2359 4850
35	1.0446 9260	1.0913 2309	1.1399 7293	1.1907 2689	1.2436 7318
36	1.0459 9847	1.0940 5140	1.1442 4783	1.1966 8052	1.2514 4614
37	1.0473 0596	1.0967 8653	1.1485 3876	1.2026 6393	1.2592 6767
38	1.0486 1510	1.0995 2850	1.1528 4578	1.2086 7725	1.2671 3810
39	1.0499 2586	1.1022 7732	1.1571 6895	1.2147 2063	1.2750 5771
40	1.0512 3827	1.1050 3301	1.1615 0834	1.2207 9424	1.2830 2682
41	1.0525 5232	1.1077 9559	1.1658 6399	1.2268 9821	1.2910 4574
42	1.0538 6801	1.1105 6508	1.1702 3598	1.2330 3270	1.2991 1477
43	1.0551 8535	1.1133 4149	1.1746 2437	1.2391 9786	1.3072 3424
44	1.0565 0433	1.1161 2485	1.1790 2921	1.2453 9385	1.3154 0446
45	1.0578 2496	1.1189 1516	1.1834 5057	1.2516 2082	1.3236 2573
46	1.0591 4724	1.1217 1245	1.1878 8851	1.2578 7892	1.3318 9839
47	1.0604 7117	1.1245 1673	1.1923 4309	1.2641 6832	1.3402 2276
48	1.0617 9676	1.1273 2802	1.1968 1438	1.2704 8916	1.3485 9915
49	1.0631 2401	1.1301 4634	1.2013 0243	1.2768 4161	1.3570 2790
50	1.0644 5291	1.1329 7171	1.2058 0732	1.2832 2581	1.3655 0932

$(1 + i)^n$

n	¾%	1%	1⅛%	1¼%	1⅜%
1	1.0075	1.01	1.0112 5	1.0125	1.0137 5
2	1.0150 5625	1.0201	1.0226 2656	1.0251 5625	1.0276 8906
3	1.0226 6917	1.0303 01	1.0341 3111	1.0379 7070	1.0418 1979
4	1.0303 3919	1.0406 0401	1.0457 6509	1.0509 4534	1.0561 4481
5	1.0380 6673	1.0510 1005	1.0575 2994	1.0640 8215	1.0706 6680
6	1.0458 5224	1.0615 2015	1.0694 2716	1.0773 8318	1.0853 8847
7	1.0536 9613	1.0721 3535	1.0814 5821	1.0908 5047	1.1003 1256
8	1.0615 9885	1.0828 5671	1.0936 2462	1.1044 8610	1.1154 4186
9	1.0695 6084	1.0936 8527	1.1059 2789	1.1182 9218	1.1307 7918
10	1.0775 8255	1.1046 2213	1.1183 6958	1.1322 7083	1.1463 2740
11	1.0856 6441	1.1156 6835	1.1309 5124	1.1464 2422	1.1620 8940
12	1.0938 0690	1.1268 2503	1.1436 7444	1.1607 5452	1.1780 6813
13	1.1020 1045	1.1380 9328	1.1565 4078	1.1752 6395	1.1942 6656
14	1.1102 7553	1.1494 7421	1.1695 5186	1.1899 5475	1.2106 8773
15	1.1186 0259	1.1609 6896	1.1827 0932	1.2048 2918	1.2273 3469
16	1.1269 9211	1.1725 7864	1.1960 1480	1.2198 8955	1.2442 1054
17	1.1354 4455	1.1843 0443	1.2094 6997	1.2351 3817	1.2613 1843
18	1.1439 6039	1.1961 4748	1.2230 7650	1.2505 7739	1.2786 6156
19	1.1525 4009	1.2081 0895	1.2368 3611	1.2662 0961	1.2962 4316
20	1.1611 8414	1.2201 9004	1.2507 5052	1.2820 3723	1.3140 6650
21	1.1698 9302	1.2323 9194	1.2648 2146	1.2980 6270	1.3321 3492
22	1.1786 6722	1.2447 1586	1.2790 5071	1.3142 8848	1.3504 5177
23	1.1875 0723	1.2571 6302	1.2934 4003	1.3307 1709	1.3690 2048
24	1.1964 1353	1.2697 3465	1.3079 9123	1.3473 5105	1.3878 4451
25	1.2053 8663	1.2824 3200	1.3227 0613	1.3641 9294	1.4069 2738
26	1.2144 2703	1.2952 5631	1.3375 8657	1.3812 4535	1.4262 7263
27	1.2235 3523	1.3082 0888	1.3526 3442	1.3985 1092	1.4458 8388
28	1.2327 1175	1.3212 9097	1.3678 5156	1.4159 9230	1.4657 6478
29	1.2419 5709	1.3345 0388	1.3832 3989	1.4336 9221	1.4859 1905
30	1.2512 7176	1.3478 4892	1.3988 0134	1.4516 1336	1.5063 5043
31	1.2606 5630	1.3613 2740	1.4145 3785	1.4697 5853	1.5270 6275
32	1.2701 1122	1.3749 4068	1.4304 5140	1.4881 3051	1.5480 5986
33	1.2796 3706	1.3886 9009	1.4465 4398	1.5067 3214	1.5693 4569
34	1.2892 3434	1.4025 7699	1.4628 1760	1.5255 6629	1.5909 2419
35	1.2989 0359	1.4166 0276	1.4792 7430	1.5446 3587	1.6127 9940
36	1.3086 4537	1.4307 6878	1.4959 1613	1.5639 4382	1.6349 7539
37	1.3184 6021	1.4450 7647	1.5127 4519	1.5834 9312	1.6574 5630
38	1.3283 4866	1.4595 2724	1.5297 6357	1.6032 8678	1.6802 4633
39	1.3383 1128	1.4741 2251	1.5469 7341	1.6233 2787	1.7033 4971
40	1.3483 4861	1.4888 6373	1.5643 7687	1.6436 1946	1.7267 7077
41	1.3584 6123	1.5037 5237	1.5819 7611	1.6641 6471	1.7505 1387
42	1.3686 4969	1.5187 8989	1.5997 7334	1.6849 6677	1.7745 8343
43	1.3789 1456	1.5339 7779	1.6177 7079	1.7060 2885	1.7989 8396
44	1.3892 5642	1.5493 1757	1.6359 7071	1.7273 5421	1.8237 1999
45	1.3996 7584	1.5648 1075	1.6543 7538	1.7489 4614	1.8487 9614
46	1.4101 7341	1.5804 5885	1.6729 8710	1.7708 0797	1.8742 1708
47	1.4207 4971	1.5962 6344	1.6918 0821	1.7929 4306	1.8999 8757
48	1.4314 0533	1.6122 2608	1.7108 4105	1.8153 5485	1.9261 1240
49	1.4421 4087	1.6283 4834	1.7300 8801	1.8380 4679	1.9525 9644
50	1.4529 5693	1.6446 3182	1.7495 5150	1.8610 2237	1.9794 4464

$(1 + i)^n$

n	1½%	1⅝%	1¾%	1⅞%	2%
1	1.015	1.0162 5	1.0175	1.0187 5	1.02
2	1.0302 25	1.0327 6406	1.0353 0625	1.0378 5156	1.0404
3	1.0456 7838	1.0495 4648	1.0534 2411	1.0573 1128	1.0612 08
4	1.0613 6355	1.0666 0161	1.0718 5903	1.0771 3587	1.0824 3216
5	1.0772 8400	1.0839 3388	1.0906 1656	1.0973 3216	1.1040 8080
6	1.0934 4326	1.1015 4781	1.1097 0235	1.1179 0714	1.1261 6242
7	1.1098 4491	1.1194 4796	1.1291 2215	1.1388 6790	1.1486 8567
8	1.1264 9259	1.1376 3899	1.1488 8178	1.1602 2167	1.1716 5938
9	1.1433 8998	1.1561 2563	1.1689 8721	1.1819 7583	1.1950 9257
10	1.1605 4083	1.1749 1267	1.1894 4449	1.2041 3788	1.2189 9442
11	1.1779 4894	1.1940 0500	1.2102 5977	1.2267 1546	1.2433 7431
12	1.1956 1817	1.2134 0758	1.2314 3931	1.2497 1638	1.2682 4179
13	1.2135 5244	1.2331 2545	1.2529 8950	1.2731 4856	1.2936 0663
14	1.2317 5573	1.2531 6374	1.2749 1682	1.2970 2009	1.3194 7876
15	1.2502 3207	1.2735 2765	1.2972 2786	1.3213 3922	1.3458 6834
16	1.2689 8555	1.2942 2248	1.3199 2935	1.3461 1433	1.3727 8571
17	1.2880 2033	1.3152 5359	1.3430 2811	1.3713 5398	1.4002 4142
18	1.3073 4064	1.3366 2646	1.3665 3111	1.3970 6686	1.4282 4625
19	1.3269 5075	1.3583 4664	1.3904 4540	1.4232 6187	1.4568 1117
20	1.3468 5501	1.3804 1977	1.4147 7820	1.4499 4803	1.4859 4740
21	1.3670 5783	1.4028 5160	1.4395 3681	1.4771 3455	1.5156 6634
22	1.3875 6370	1.4256 4793	1.4647 2871	1.5048 3082	1.5459 7967
23	1.4083 7715	1.4488 1471	1.4903 6146	1.5330 4640	1.5768 9926
24	1.4295 0281	1.4723 5795	1.5164 4279	1.5617 9102	1.6084 3725
25	1.4509 4535	1.4962 8377	1.5429 8054	1.5910 7460	1.6406 0599
26	1.4727 0953	1.5205 9838	1.5699 8269	1.6209 0725	1.6734 1811
27	1.4948 0018	1.5453 0810	1.5974 5739	1.6512 9926	1.7068 8648
28	1.5172 2218	1.5704 1936	1.6254 1290	1.6822 6112	1.7410 2421
29	1.5399 8051	1.5959 3868	1.6538 5762	1.7138 0352	1.7758 4469
30	1.5630 8022	1.6218 7268	1.6828 0013	1.7459 3734	1.8113 6158
31	1.5865 2642	1.6482 2811	1.7122 4913	1.7786 7366	1.8475 8882
32	1.6103 2432	1.6750 1182	1.7422 1349	1.8120 2379	1.8845 4059
33	1.6344 7918	1.7022 3076	1.7727 0223	1.8459 9924	1.9222 3140
34	1.6589 9637	1.7298 9201	1.8037 2452	1.8806 1172	1.9606 7603
35	1.6838 8132	1.7580 0275	1.8352 8970	1.9158 7319	1.9998 8955
36	1.7091 3954	1.7865 7030	1.8674 0727	1.9517 9582	2.0398 8734
37	1.7347 7663	1.8156 0207	1.9000 8689	1.9883 9199	2.0806 8509
38	1.7607 9828	1.8451 0560	1.9333 3841	2.0256 7434	2.1222 9879
39	1.7872 1025	1.8750 8857	1.9671 7184	2.0636 5573	2.1647 4477
40	1.8140 1841	1.9055 5875	2.0015 9734	2.1023 4928	2.2080 3966
41	1.8412 2868	1.9365 2408	2.0366 2530	2.1417 6833	2.2522 0046
42	1.8688 4712	1.9679 9260	2.0722 6624	2.1819 2648	2.2972 4447
43	1.8968 7982	1.9999 7248	2.1085 3090	2.2228 3760	2.3431 8936
44	1.9253 3302	2.0324 7203	2.1454 3019	2.2645 1581	2.3900 5314
45	1.9542 1301	2.0654 9970	2.1829 7522	2.3069 7548	2.4378 5421
46	1.9835 2621	2.0990 6407	2.2211 7728	2.3502 3127	2.4866 1129
47	2.0132 7910	2.1331 7387	2.2600 4789	2.3942 9811	2.5363 4351
48	2.0434 7829	2.1678 3794	2.2995 9872	2.4391 9120	2.5870 7039
49	2.0741 3046	2.2030 6531	2.3398 4170	2.4849 2603	2.6388 1179
50	2.1052 4242	2.2388 6512	2.3807 8893	2.5315 1839	2.6915 8803

$(1+i)^n$

n	2⅛%	2¼%	2⅜%	2½%	2¾%
1	1.0212 5	1.0225	1.0237 5	1.025	1.0275
2	1.0429 5156	1.0455 0625	1.0480 6406	1.0506 25	1.0557 5625
3	1.0651 1428	1.0690 3014	1.0729 5558	1.0768 9063	1.0847 8955
4	1.0877 4796	1.0930 8332	1.0984 3828	1.1038 1289	1.1146 2126
5	1.1108 6261	1.1176 7769	1.1245 2619	1.1314 0821	1.1452 7334
6	1.1344 6844	1.1428 2544	1.1512 3369	1.1596 9342	1.1767 6836
7	1.1585 7589	1.1685 3901	1.1785 7549	1.1886 8575	1.2091 2949
8	1.1831 9563	1.1948 3114	1.2065 6665	1.2184 0290	1.2423 8055
9	1.2083 3854	1.2217 1484	1.2352 2261	1.2488 6297	1.2765 4602
10	1.2340 1573	1.2492 0343	1.2645 5915	1.2800 8454	1.3116 5103
11	1.2602 3856	1.2773 1050	1.2945 9243	1.3120 8666	1.3477 2144
12	1.2870 1863	1.3060 4999	1.3253 3900	1.3448 8882	1.3847 8378
13	1.3143 6778	1.3354 3611	1.3568 1580	1.3785 1104	1.4228 6533
14	1.3422 9809	1.3654 8343	1.3890 4017	1.4129 7382	1.4619 9413
15	1.3708 2193	1.3962 0680	1.4220 2988	1.4482 9817	1.5021 9896
16	1.3999 5189	1.4276 2146	1.4558 0309	1.4845 0562	1.5435 0944
17	1.4297 0087	1.4597 4294	1.4903 7841	1.5216 1826	1.5859 5595
18	1.4600 8202	1.4925 8716	1.5257 7490	1.5596 5872	1.6295 6973
19	1.4911 0876	1.5261 7037	1.5620 1205	1.5986 5019	1.6743 8290
20	1.5227 9482	1.5605 0920	1.5991 0984	1.6386 1644	1.7204 2843
21	1.5551 5421	1.5956 2066	1.6370 8870	1.6795 8185	1.7677 4021
22	1.5882 0124	1.6315 2212	1.6759 6955	1.7215 7140	1.8163 5307
23	1.6219 5051	1.6682 3137	1.7157 7383	1.7646 1068	1.8663 0278
24	1.6564 1696	1.7057 6658	1.7565 2346	1.8087 2595	1.9176 2610
25	1.6916 1582	1.7441 4632	1.7982 4089	1.8539 4410	1.9703 6082
26	1.7275 6266	1.7833 8962	1.8409 4911	1.9002 9270	2.0245 4575
27	1.7642 7336	1.8235 1588	1.8846 7165	1.9478 0002	2.0802 2075
28	1.8017 6417	1.8645 4499	1.9294 3261	1.9964 9502	2.1374.2682
29	1.8400 5166	1.9064 9725	1.9752 5663	2.0464 0739	2.1962 0606
30	1.8791 5276	1.9493 9344	2.0221 6898	2.0975 6758	2.2566 0173
31	1.9190 8476	1.9932 5479	2.0701 9549	2.1500 0677	2.3186 5828
32	1.9598 6531	2.0381 0303	2.1193 6263	2.2037 5694	2.3824 2138
33	2.0015 1245	2.0839 6034	2.1696 9749	2.2588 5086	2.4479 3797
34	2.0440 4458	2.1308 4945	2.2212 2781	2.3153 2213	2.5152 5626
35	2.0874 8053	2.1787 9356	2.2739 8197	2.3732 0519	2.5844 2581
36	2.1318 3949	2.2278 1642	2.3279 8904	2.4325 3532	2.6554 9752
37	2.1771 4108	2.2779 4229	2.3832 7878	2.4933 4870	2.7285 2370
38	2.2234 0533	2.3291 9599	2.4398 8165	2.5556 8242	2.8035 5810
39	2.2706 5269	2.3816 0290	2.4978 2884	2.6195 7448	2.8806 5595
40	2.3189 0406	2.4351 8897	2.5571 5228	2.6850 6384	2.9598 7399
41	2.3681 8077	2.4899 8072	2.6178 8464	2.7521 9043	3.0412 7052
42	2.4185 0462	2.5460 0528	2.6800 5940	2.8209 9520	3.1249 0546
43	2.4698 9784	2.6032 9040	2.7437 1081	2.8915 2008	3.2108 4036
44	2.5223 8317	2.6618 6444	2.8088 7395	2.9638 0808	3.2991 3847
45	2.5759 8381	2.7217 5639	2.8755 8470	3.0379 0328	3.3898 6478
46	2.6307 2347	2.7829 9590	2.9438 7984	3.1138 5086	3.4830 8606
47	2.6866 2634	2.8456 1331	3.0137 9699	3.1916 9713	3.5788 7093
48	2.7437 1715	2.9096 3961	3.0853 7466	3.2714 8956	3.6772 8988
49	2.8020 2114	2.9751 0650	3.1586 5231	3.3532 7680	3.7784 1535
50	2.8615 6409	3.0420 4640	3.2336 7030	3.4371 0872	3.8823 2177

$(1 + i)^n$

n	3%	3¼%	3½%	3¾%	4%
1	1.03	1.0325	1.035	1.0375	1.04
2	1.0609	1.0660 5625	1.0712 25	1.0764 0625	1.0816
3	1.0927 27	1.1007 0308	1.1087 1788	1.1167 7148	1.1248 64
4	1.1255 0881	1.1364 7593	1.1475 2300	1.1586 5042	1.1698 5856
5	1.1592 7407	1.1734 1140	1.1876 8631	1.2020 9981	1.2166 5290
6	1.1940 5230	1.2115 4727	1.2292 5533	1.2471 7855	1.2653 1902
7	1.2298 7387	1.2509 2255	1.2722 7926	1.2939 4774	1.3159 3178
8	1.2667 7008	1.2915 7754	1.3168 0904	1.3424 7078	1.3685 6905
9	1.3047 7318	1.3335 5381	1.3628 9735	1.3928 1344	1.4233 1181
10	1.3439 1638	1.3768 9430	1.4105 9876	1.4450 4394	1.4802 4428
11	1.3842 3387	1.4216 4337	1.4599 6972	1.4992 3309	1.5394 5406
12	1.4257 6089	1.4678 4678	1.5110 6866	1.5554 5433	1.6010 3222
13	1.4685 3371	1.5155 5180	1.5639 5606	1.6137 8387	1.6650 7351
14	1.5125 8972	1.5648 0723	1.6186 9452	1.6743 0076	1.7316 7645
15	1.5579 6742	1.6156 6347	1.6753 4883	1.7370 8704	1.8009 4351
16	1.6047 0644	1.6681 7253	1.7339 8604	1.8022 2781	1.8729 8125
17	1.6528 4763	1.7223 8814	1.7946 7555	1.8698 1135	1.9479 0050
18	1.7024 3306	1.7783 6575	1.8574 8920	1.9399 2927	2.0258 1652
19	1.7535 0605	1.8361 3264	1.9225 0132	2.0126 7662	2.1068 4918
20	1.8061 1123	1.8958 3792	1.9897 8886	2.0881 5200	2.1911 2314
21	1.8602 9457	1.9574 5266	2.0594 3147	2.1664 5770	2.2787 6807
22	1.9161 0341	2.0210 6987	2.1315 1158	2.2476 9986	2.3699 1879
23	1.9735 8651	2.0867 5464	2.2061 1448	2.3319 8860	2.4647 1554
24	2.0327 9411	2.1545 7416	2.2833 2849	2.4194 3818	2.5633 0416
25	2.0937 7793	2.2245 9782	2.3632 4498	2.5101 6711	2.6658 3633
26	2.1565 9127	2.2968 9725	2.4459 5856	2.6042 9838	2.7724 6978
27	2.2212 8901	2.3715 4641	2.5315 6711	2.7019 5956	2.8833 6858
28	2.2879 2768	2.4486 2167	2.6201 7196	2.8032 8305	2.9987 0332
29	2.3565 6551	2.5282 0188	2.7118 7798	2.9084 0616	3.1186 5145
30	2.4272 6247	2.6103 6844	2.8067 9370	3.0174 7139	3.2433 9751
31	2.5000 8035	2.6952 0541	2.9050 3148	3.1306 2657	3.3731 3341
32	2.5750 8276	2.7827 9959	3.0067 0759	3.2480 2507	3.5080 5875
33	2.6523 3524	2.8732 4058	3.1119 4235	3.3698 2601	3.6483 8110
34	2.7319 0530	2.9666 2089	3.2208 6033	3.4961 9448	3.7943 1634
35	2.8138 6245	3.0630 3607	3.3335 9045	3.6273 0178	3.9460 8899
36	2.8982 7833	3.1625 8475	3.4502 6611	3.7633 2559	4.1039 3255
37	2.9852 2668	3.2653 6875	3.5710 2543	3.9044 5030	4.2680 8986
38	3.0747 8348	3.3714 9323	3.6960 1132	4.0508 6719	4.4388 1345
39	3.1670 2698	3.4810 6676	3.8253 7171	4.2027 7471	4.6163 6599
40	3.2620 3779	3.5942 0143	3.9592 5972	4.3603 7876	4.8010 2063
41	3.3598 9893	3.7110 1298	4.0978 3381	4.5238 9296	4.9930 6145
42	3.4606 9589	3.8316 2090	4.2412 5799	4.6935 3895	5.1927 8391
43	3.5645 1677	3.9561 4858	4.3897 0202	4.8695 4666	5.4004 9527
44	3.6714 5227	4.0847 2341	4.5433 4160	5.0521 5466	5.6165 1508
45	3.7815 9584	4.2174 7692	4.7023 5855	5.2416 1046	5.8411 7568
46	3.8950 4372	4.3545 4492	4.8669 4110	5.4381 7085	6.0748 2271
47	4.0118 9503	4.4960 6763	5.0372 8404	5.6421 0226	6.3178 1562
48	4.1322 5188	4.6421 8983	5.2135 8898	5.8536 8109	6.5705 2824
49	4.2562 1944	4.7930 6100	5.3960 6459	6.0731 9413	6.8333 4937
50	4.3839 0602	4.9488 3548	5.5849 2686	6.3009 3891	7.1066 8335

$(1 + i)^n$

n	4¼%	4½%	4¾%	5%	5½%
1	1.0425	1.045	1.0475	1.05	1.055
2	1.0868 0625	1.0920 25	1.0972 5625	1.1025	1.1130 25
3	1.1329 9552	1.1411 6613	1.1493 7592	1.1576 25	1.1742 4138
4	1.1811 4783	1.1925 1860	1.2039 7128	1.2155 0625	1.2388 2465
5	1.2313 4661	1.2461 8194	1.2611 5991	1.2762 8156	1.3069 6001
6	1.2836 7884	1.3022 6012	1.3210 6501	1.3400 9564	1.3788 4281
7	1.3382 3519	1.3608 6183	1.3838 1560	1.4071 0042	1.4546 7916
8	1.3951 1018	1.4221 0061	1.4495 4684	1.4774 5544	1.5346 8651
9	1.4544 0237	1.4860 9514	1.5184 0031	1.5513 2822	1.6190 9427
10	1.5162 1447	1.5529 6942	1.5905 2433	1.6288 9463	1.7081 4446
11	1.5806 5358	1.6228 5305	1.6660 7423	1.7103 3936	1.8020 9240
12	1.6478 3136	1.6958 8143	1.7452 1276	1.7958 5633	1.9012 0749
13	1.7178 6419	1.7721 9610	1.8281 1037	1.8856 4914	2.0057 7390
14	1.7908 7342	1.8519 4492	1.9149 4561	1.9799 3160	2.1160 9146
15	1.8669 8554	1.9352 8244	2.0059 0552	2.0789 2818	2.2324 7649
16	1.9463 3243	2.0223 7015	2.1011 8604	2.1828 7459	2.3552 6270
17	2.0290 5156	2.1133 7681	2.2009 9237	2.2920 1832	2.4848 0215
18	2.1152 8625	2.2084 7877	2.3055 3951	2.4066 1923	2.6214 6627
19	2.2051 8591	2.3078 6031	2.4150 5264	2.5269 5020	2.7656 4691
20	2.2989 0631	2.4117 1402	2.5297 6764	2.6532 9771	2.9177 5749
21	2.3966 0983	2.5202 4116	2.6499 3160	2.7859 6259	3.0782 3415
22	2.4984 6575	2.6336 5201	2.7758 0335	2.9252 6072	3.2475 3703
23	2.6046 5054	2.7521 6635	2.9076 5401	3.0715 2376	3.4261 5157
24	2.7153 4819	2.8760 1383	3.0457 6758	3.2250 9994	3.6145 8990
25	2.8307 5049	3.0054 3446	3.1904 4154	3.3863 5494	3.8133 9235
26	2.9510 5739	3.1406 7901	3.3419 8751	3.5556 7269	4.0231 2893
27	3.0764 7732	3.2820 0956	3.5007 3192	3.7334 5632	4.2444 0102
28	3.2072 2761	3.4296 9998	3.6670 1668	3.9201 2914	4.4778 4307
29	3.3435 3478	3.5840 3649	3.8411 9998	4.1161 3560	4.7241 2444
30	3.4856 3501	3.7453 1813	4.0236 5698	4.3219 4238	4.9839 5129
31	3.6337 7450	3.9138 5745	4.2147 8068	4.5380 3949	5.2580 6861
32	3.7882 0992	4.0899 8104	4.4149 8276	4.7649 4147	5.5472 6238
33	3.9492 0884	4.2740 3018	4.6246 9445	5.0031 8854	5.8523 6181
34	4.1170 5021	4.4663 6154	4.8443 6743	5.2533 4797	6.1742 4171
35	4.2920 2485	4.6673 4781	5.0744 7488	5.5160 1537	6.5138 2501
36	4.4744 3590	4.8773 7846	5.3155 1244	5.7918 1614	6.8720 8538
37	4.6645 9943	5.0968 6049	5.5679 9928	6.0814 0694	7.2500 5008
38	4.8628 4491	5.3262 1921	5.8324 7925	6.3854 7729	7.6488 0283
39	5.0695 1581	5.5658 9908	6.1095 2201	6.7047 5115	8.0694 8699
40	5.2849 7024	5.8163 6454	6.3997 2431	7.0399 8871	8.5133 0877
41	5.5095 8147	6.0781 0094	6.7037 1121	7.3919 8815	8.9815 4076
42	5.7437 3868	6.3516 1548	7.0221 3750	7.7615 8756	9.4755 2550
43	5.9878 4758	6.6374 3818	7.3556 8903	8.1496 6693	9.9966 7940
44	6.2423 3110	6.9361 2290	7.7050 8426	8.5571 5028	10.5464 9677
45	6.5076 3017	7.2482 4843	8.0710 7576	8.9850 0779	11.1265 5409
46	6.7842 0445	7.5744 1961	8.4544 5186	9.4342 5818	11.7385 1456
47	7.0725 3314	7.9152 6849	8.8560 3832	9.9059 7109	12.3841 3287
48	7.3731 1580	8.2714 5557	9.2767 0014	10.4012 6965	13.0652 6017
49	7.6864 7322	8.6436 7107	9.7173 4340	10.9213 3313	13.7838 4948
50	8.0131 4834	9.0326 3627	10.1789 1721	11.4673 9979	14.5419 6120

$(1+i)^n$

n	6%	6½%	7%	8%	9%
1	1.06	1.065	1.07	1.08	1.09
2	1.1236	1.1342 25	1.1449	1.1664	1.1881
3	1.1910 16	1.2079 4963	1.2250 43	1.2597 12	1.2950 29
4	1.2624 7696	1.2864 6635	1.3107 9601	1.3604 8896	1.4115 8161
5	1.3382 2558	1.3700 8666	1.4025 5173	1.4693 2808	1.5386 2395
6	1.4185 1911	1.4591 4230	1.5007 3035	1.5868 7432	1.6771 0011
7	1.5036 3026	1.5539 8655	1.6057 8148	1.7138 2427	1.8280 3912
8	1.5938 4807	1.6549 9567	1.7181 8618	1.8509 3021	1.9925 6264
9	1.6894 7896	1.7625 7039	1.8384 5921	1.9990 0463	2.1718 9328
10	1.7908 4770	1.8771 3747	1.9671 5136	2.1589 2500	2.3673 6367
11	1.8982 9856	1.9991 5140	2.1048 5195	2.3316 3900	2.5804 2641
12	2.0121 9647	2.1290 9624	2.2521 9159	2.5181 7012	2.8126 6478
13	2.1329 2826	2.2674 8750	2.4098 4500	2.7196 2373	3.0658 0461
14	2.2609 0396	2.4148 7418	2.5785 3415	2.9371 9362	3.3417 2703
15	2.3965 5819	2.5718 4101	2.7590 3154	3.1721 6911	3.6424 8246
16	2.5403 5168	2.7390 1067	2.9521 6375	3.4259 4264	3.9703 0588
17	2.6927 7279	2.9170 4637	3.1588 1521	3.7000 1805	4.3276 3341
18	2.8543 3915	3.1066 5438	3.3799 3228	3.9960 1950	4.7171 2042
19	3.0255 9950	3.3085 8691	3.6165 2754	4.3157 0106	5.1416 6125
20	3.2071 3547	3.5236 4506	3.8696 8446	4.6609 5714	5.6044 1077
21	3.3995 6360	3.7526 8199	4.1405 6237	5.0338 3372	6.1088 0774
22	3.6035 3742	3.9966 0632	4.4304 0174	5.4365 4041	6.6586 0043
23	3.8197 4966	4.2563 8573	4.7405 2986	5.8714 6365	7.2578 7447
24	4.0489 3464	4.5330 5081	5.0723 6695	6.3411 8074	7.9110 8317
25	4.2918 7072	4.8276 9911	5.4274 3264	6.8484 7520	8.6230 8066
26	4.5493 8296	5.1414 9955	5.8073 5292	7.3963 5321	9.3991 5792
27	4.8223 4594	5.4756 9702	6.2138 6763	7.9880 6147	10.2450 8213
28	5.1116 8670	5.8316 1733	6.6488 3836	8.6271 0639	11.1671 3952
29	5.4183 8790	6.2106 7245	7.1142 5705	9.3172 7490	12.1721 8208
30	5.7434 9117	6.6143 6616	7.6122 5504	10.0626 5689	13.2676 7847
31	6.0881 0064	7.0442 9996	8.1451 1290	10.8676 6944	14.4617 6953
32	6.4533 8668	7.5021 7946	8.7152 7080	11.7370 8300	15.7633 2879
33	6.8405 8988	7.9898 2113	9.3253 3975	12.6760 4964	17.1820 2838
34	7.2510 2528	8.5091 5950	9.9781 1354	13.6901 3361	18.7284 1093
35	7.6860 8679	9.0622 5487	10.6765 8148	14.7853 4429	20.4139 6792
36	8.1472 5200	9.6513 0143	11.4239 4219	15.9681 7184	22.2512 2503
37	8.6360 8712	10.2786 3603	12.2236 1814	17.2456 2558	24.2538 3528
38	9.1542 5235	10.9467 4737	13.0792 7141	18.6252 7563	26.4366 8046
39	9.7035 0749	11.6582 8595	13.9948 2041	20.1152 9768	28.8159 8170
40	10.2857 1794	12.4160 7453	14.9744 5784	21.7245 2150	31.4094 2005
41	10.9028 6101	13.2231 1938	16.0226 6989	23.4624 8322	34.2362 6786
42	11.5570 3267	14.0826 2214	17.1442 5678	25.3394 8187	37.3175 3197
43	12.2504 5463	14.9979 9258	18.3443 5475	27.3666 4042	40.6761 0984
44	12.9854 8191	15.9728 6209	19.6284 5959	29.5559 7166	44.3369 5973
45	13.7646 1083	17.0110 9813	21.0024 5176	31.9204 4939	48.3272 8610
46	14.5904 8748	18.1168 1951	22.4726 2338	34.4740 8534	52.6767 4185
47	15.4659 1673	19.2944 1278	24.0457 0702	37.2320 1217	57.4176 4862
48	16.3938 7173	20.5485 4961	25.7289 0651	40.2105 7314	62.5852 3700
49	17.3775 0403	21.8842 0533	27.5299 2997	43.4274 1899	68.2179 0833
50	18.4201 5427	23.3066 7868	29.4570 2506	46.9016 1251	74.3575 2008

$$\frac{1}{(1+i)^n}$$

n	⅛%	¼%	⅜%	½%	⅝%
1	0.9987 5156	0.9975 0623	0.9962 6401	0.9950 2488	0.9937 8882
2	0.9975 0468	0.9950 1869	0.9925 4198	0.9900 7450	0.9876 1622
3	0.9962 5936	0.9925 3734	0.9888 3385	0.9851 4876	0.9814 8196
4	0.9950 1559	0.9900 6219	0.9851 3958	0.9802 4752	0.9753 8580
5	0.9937 7337	0.9875 9321	0.9814 5911	0.9753 7067	0.9693 2750
6	0.9925 3270	0.9851 3038	0.9777 9238	0.9705 1808	0.9633 0683
7	0.9912 9359	0.9826 7370	0.9741 3936	0.9656 8963	0.9573 2356
8	0.9900 5602	0.9802 2314	0.9704 9999	0.9608 8520	0.9513 7745
9	0.9888 1999	0.9777 7869	0.9668 7421	0.9561 0468	0.9454 6827
10	0.9875 8551	0.9753 4034	0.9632 6198	0.9513 4794	0.9395 9580
11	0.9863 5257	0.9729 0807	0.9596 6324	0.9466 1489	0.9337 5980
12	0.9851 2117	0.9704 8187	0.9560 7795	0.9419 0534	0.9279 6005
13	0.9838 9130	0.9680 6171	0.9525 0605	0.9372 1924	0.9221 9632
14	0.9826 6297	0.9656 4759	0.9489 4750	0.9325 5646	0.9164 6840
15	0.9814 3618	0.9632 3949	0.9454 0224	0.9279 1688	0.9107 7604
16	0.9802 1092	0.9608 3740	0.9418 7022	0.9233 0037	0.9051 1905
17	0.9789 8718	0.9584 4130	0.9383 5141	0.9187 0684	0.8994 9719
18	0.9777 6498	0.9560 5117	0.9348 4573	0.9141 3616	0.8939 1025
19	0.9765 4430	0.9536 6700	0.9313 5316	0.9095 8822	0.8883 5802
20	0.9753 2514	0.9512 8878	0.9278 7363	0.9050 6290	0.8828 4027
21	0.9741 0750	0.9489 1649	0.9244 0711	0.9005 6010	0.8773 5679
22	0.9728 9139	0.9465 5011	0.9209 5353	0.8960 7971	0.8719 0736
23	0.9716 7679	0.9441 8964	0.9175 1286	0.8916 2160	0.8664 9179
24	0.9704 6371	0.9418 3505	0.9140 8504	0.8871 8567	0.8611 0985
25	0.9692 5215	0.9394 8634	0.9106 7003	0.8827 7181	0.8557 6135
26	0.9680 4210	0.9371 4348	0.9072 6777	0.8783 7991	0.8504 4606
27	0.9668 3355	0.9348 0646	0.9038 7823	0.8740 0986	0.8451 6378
28	0.9656 2652	0.9324 7527	0.9005 0135	0.8696 6155	0.8399 1432
29	0.9644 2100	0.9301 4990	0.8971 3709	0.8653 3488	0.8346 9746
30	0.9632 1697	0.9278 3032	0.8937 8539	0.8610 2973	0.8295 1300
31	0.9620 1446	0.9255 1653	0.8904 4622	0.8567 4600	0.8243 6075
32	0.9608 1344	0.9232 0851	0.8871 1952	0.8524 8358	0.8192 4050
33	0.9596 1392	0.9209 0624	0.8838 0525	0.8482 4237	0.8141 5205
34	0.9584 1590	0.9186 0972	0.8805 0336	0.8440 2226	0.8090 9520
35	0.9572 1938	0.9163 1892	0.8772 1381	0.8398 2314	0.8040 6976
36	0.9560 2435	0.9140 3384	0.8739 3655	0.8356 4492	0.7990 7554
37	0.9548 3081	0.9117 5445	0.8706 7153	0.8314 8748	0.7941 1234
38	0.9536 3876	0.9094 8075	0.8674 1871	0.8273 5073	0.7891 7997
39	0.9524 4820	0.9072 1272	0.8641 7804	0.8232 3455	0.7842 7823
40	0.9512 5913	0.9049 5034	0.8609 4948	0.8191 3886	0.7794 0693
41	0.9500 7154	0.9026 9361	0.8577 3298	0.8150 6354	0.7745 6590
42	0.9488 8543	0.9004 4250	0.8545 2850	0.8110 0850	0.7697 5493
43	0.9477 0080	0.8981 9701	0.8513 3599	0.8069 7363	0.7649 7384
44	0.9465 1766	0.8959 5712	0.8481 5541	0.8029 5884	0.7602 2245
45	0.9453 3599	0.8937 2281	0.8449 8671	0.7989 6402	0.7555 0057
46	0.9441 5579	0.8914 9407	0.8418 2985	0.7949 8907	0.7508 0802
47	0.9429 7707	0.8892 7090	0.8386 8478	0.7910 3390	0.7461 4462
48	0.9417 9982	0.8870 5326	0.8355 5146	0.7870 9841	0.7415 1018
49	0.9406 2404	0.8848 4116	0.8324 2985	0.7831 8250	0.7369 0453
50	0.9394 4973	0.8826 3457	0.8293 1990	0.7792.8607	0.7323 2748

$$\frac{1}{(1+i)^n}$$

n	¾%	1%	1⅛%	1¼%	1⅜%
1	0.9925 5583	0.9900 9901	0.9888 7515	0.9876 5432	0.9864 3650
2	0.9851 6708	0.9802 9605	0.9778 7407	0.9754 6106	0.9730 5696
3	0.9778 3333	0.9705 9015	0.9669 9537	0.9634 1833	0.9598 5890
4	0.9705 5417	0.9609 8034	0.9562 3770	0.9515 2428	0.9468 3986
5	0.9633 2920	0.9514 6569	0.9455 9970	0.9397 7706	0.9339 9739
6	0.9561 5802	0.9420 4524	0.9350 8005	0.9281 7488	0.9213 2912
7	0.9490 4022	0.9327 1805	0.9246 7743	0.9167 1593	0.9088 3267
8	0.9419 7540	0.9234 8322	0.9143 9054	0.9053 9845	0.8965 0571
9	0.9349 6318	0.9143 3982	0.9042 1808	0.8942 2069	0.8843 4596
10	0.9280 0315	0.9052 8695	0.8941 5880	0.8831 8093	0.8723 5113
11	0.9210 9494	0.8963 2372	0.8842 1142	0.8722 7746	0.8605 1899
12	0.9142 3815	0.8874 4923	0.8743 7470	0.8615 0860	0.8488 4734
13	0.9074 3241	0.8786 6260	0.8646 4742	0.8508 7269	0.8373 3400
14	0.9006 7733	0.8699 6297	0.8550 2835	0.8403 6809	0.8259 7682
15	0.8939 7254	0.8613 4947	0.8455 1629	0.8299 9318	0.8147 7368
16	0.8873 1766	0.8528 2126	0.8361 1005	0.8197 4635	0.8037 2250
17	0.8807 1231	0.8443 7749	0.8268 0846	0.8096 2602	0.7928 2120
18	0.8741 5614	0.8360 1731	0.8176 1034	0.7996 3064	0.7820 6777
19	0.8676 4878	0.8277 3992	0.8085 1455	0.7897 5866	0.7714 6020
20	0.8611 8985	0.8195 4447	0.7995 1995	0.7800 0855	0.7609 9649
21	0.8547 7901	0.8114 3017	0.7906 2542	0.7703 7881	0.7506 7472
22	0.8484 1589	0.8033 9621	0.7818 2983	0.7608 6796	0.7404 9294
23	0.8421 0014	0.7954 4179	0.7731 3210	0.7514 7453	0.7304 4926
24	0.8358 3140	0.7875 6613	0.7645 3112	0.7421 9707	0.7205 4181
25	0.8296 0933	0.7797 6844	0.7560 2583	0.7330 3414	0.7107 6874
26	0.8234 3358	0.7720 4796	0.7476 1516	0.7239 8434	0.7011 2823
27	0.8173 0380	0.7644 0392	0.7392 9806	0.7150 4626	0.6916 1847
28	0.8112 1966	0.7568 3557	0.7310 7348	0.7062 1853	0.6822 3771
29	0.8051 8080	0.7493 4215	0.7229 4040	0.6974 9978	0.6729 8417
30	0.7991 8690	0.7419 2292	0.7148 9780	0.6888 8867	0.6638 5615
31	0.7932 3762	0.7345 7715	0.7069 4467	0.6803 8387	0.6548 5194
32	0.7873 3262	0.7273 0411	0.6990 8002	0.6719 8407	0.6459 6985
33	0.7814 7158	0.7201 0307	0.6913 0287	0.6636 8797	0.6372 0824
34	0.7756 5418	0.7129 7334	0.6836 1223	0.6554 9429	0.6285 6546
35	0.7698 8008	0.7059 1420	0.6760 0715	0.6474 0177	0.6200 3991
36	0.7641 4896	0.6989 2495	0.6684 8667	0.6394 0916	0.6116 3000
37	0.7584 6051	0.6920 0490	0.6610 4986	0.6315 1522	0.6033 3416
38	0.7528 1440	0.6851 5337	0.6536 9578	0.6237 1873	0.5951 5083
39	0.7472 1032	0.6783 6967	0.6464 2352	0.6160 1850	0.5870 7850
40	0.7416 4796	0.6716 5314	0.6392 3216	0.6084 1334	0.5791 1566
41	0.7361 2701	0.6650 0311	0.6321 2080	0.6009 0206	0.5712 6083
42	0.7306 4716	0.6584 1892	0.6250 8855	0.5934 8352	0.5635 1253
43	0.7252 0809	0.6518 9992	0.6181 3454	0.5861 5656	0.5558 6933
44	0.7198 0952	0.6454 4546	0.6112 5789	0.5789 2006	0.5483 2979
45	0.7144 5114	0.6390 5492	0.6044 5774	0.5717 7290	0.5408 9252
46	0.7091 3264	0.6327 2764	0.5977 3324	0.5647 1397	0.5335 5612
47	0.7038 5374	0.6264 6301	0.5910 8355	0.5577 4219	0.5263 1923
48	0.6986 1414	0.6202 6041	0.5845 0784	0.5508 5649	0.5191 8050
49	0.6934 1353	0.6141 1921	0.5780 0528	0.5440 5579	0.5121 3860
50	0.6882 5165	0.6080 3882	0.5715 7506	0.5373 3905	0.5051 9220

$$\frac{1}{(1+i)^n}$$

n	1½%	1⅝%	1¾%	1⅞%	2%
1	0.9852 2167	0.9840 0984	0.9828 0098	0.9815 9509	0.9803 9216
2	0.9706 6175	0.9682 7537	0.9658 9777	0.9635 2892	0.9611 6878
3	0.9563 1699	0.9527 9249	0.9492 8528	0.9457 9526	0.9423 2233
4	0.9421 8423	0.9375 5718	0.9329 5851	0.9283 8799	0.9238 4543
5	0.9282 6033	0.9225 6549	0.9169 1254	0.9113 0109	0.9057 3081
6	0.9145 4219	0.9078 1352	0.9011 4254	0.8945 2868	0.8879 7138
7	0.9010 2679	0.8932 9744	0.8856 4378	0.8780 6496	0.8705 6018
8	0.8877 1112	0.8790 1347	0.8704 1157	0.8619 0426	0.8534 9037
9	0.8745 9224	0.8649 5791	0.8554 4135	0.8460 4099	0.8367 5527
10	0.8616 6723	0.8511 2709	0.8407 2860	0.8304 6968	0.8202 4830
11	0.8489 3323	0.8375 1743	0.8262 6889	0.8151 8496	0.8042 6304
12	0.8363 8742	0.8241 2539	0.8120 5788	0.8001 8156	0.7884 9318
13	0.8240 2702	0.8109 4750	0.7980 9128	0.7854 5429	0.7730 3253
14	0.8118 4928	0.7979 8032	0.7843 6490	0.7709 9808	0.7578 7502
15	0.7998 5150	0.7852 2048	0.7708 7459	0.7568 0793	0.7430 1473
16	0.7880 3104	0.7726 6468	0.7576 1631	0.7428 7895	0.7284 4581
17	0.7763 8526	0.7603 0965	0.7445 8605	0.7292 0633	0.7141 6256
18	0.7649 1159	0.7481 5218	0.7317 7990	0.7157 8536	0.7001 5937
19	0.7536 0747	0.7361 8911	0.7191 9401	0.7026 1139	0.6864 3076
20	0.7424 7042	0.7244 1732	0.7068 2458	0.6896 7989	0.6729 7133
21	0.7314 9795	0.7128 3378	0.6946 6789	0.6769 8640	0.6597 7582
22	0.7206 8763	0.7014 3545	0.6827 2028	0.6645 2653	0.6468 3904
23	0.7100 3708	0.6902 1938	0.6709 7817	0.6522 9598	0.6341 5592
24	0.6995 4392	0.6791 8267	0.6594 3800	0.6402 9053	0.6217 2149
25	0.6892 0583	0.6683 2243	0.6480 9632	0.6285 0604	0.6095 3087
26	0.6790 2052	0.6576 3584	0.6369 4970	0.6169 3844	0.5975 7928
27	0.6689 8574	0.6471 2014	0.6259 9479	0.6055 8375	0.5858 6204
28	0.6590 9925	0.6367 7259	0.6152 2829	0.5944 3804	0.5743 7455
29	0.6493 5887	0.6265 9049	0.6046 4697	0.5834 9746	0.5631 1231
30	0.6397 6243	0.6165 7121	0.5942 4764	0.5727 5824	0.5520 7089
31	0.6303 0781	0.6067 1214	0.5840 2716	0.5622 1668	0.5412 4597
32	0.6209 9292	0.5970 1071	0.5739 8247	0.5518 6913	0.5306 3330
33	0.6118 1568	0.5874 6442	0.5641 1053	0.5417 1203	0.5202 2873
34	0.6027 7407	0.5780 7077	0.5544 0839	0.5317 4187	0.5100 2817
35	0.5938 6608	0.5688 2732	0.5448 7311	0.5219 5521	0.5000 2761
36	0.5850 8974	0.5597 3168	0.5355 0183	0.5123 4867	0.4902 2315
37	0.5764 4309	0.5507 8148	0.5262 9172	0.5029 1894	0.4806 1093
38	0.5679 2423	0.5419 7440	0.5172 4002	0.4936 6277	0.4711 8719
39	0.5595 3126	0.5333 0814	0.5083 4400	0.4845 7695	0.4619 4822
40	0.5512 6232	0.5247 8046	0.4996 0098	0.4756 5836	0.4528 9042
41	0.5431 1559	0.5163 8914	0.4910 0834	0.4669 0391	0.4440 1021
42	0.5350 8925	0.5081 3199	0.4825 6348	0.4583 1058	0.4353 0413
43	0.5271 8153	0.5000 0688	0.4742 6386	0.4498 7542	0.4267 6875
44	0.5193 9067	0.4920 1169	0.4661 0699	0.4415 9550	0.4184 0074
45	0.5117 1494	0.4841 4434	0.4580 9040	0.4334 6798	0.4101 9680
46	0.5041 5265	0.4764 0280	0.4502 1170	0.4254 9004	0.4021 5373
47	0.4967 0212	0.4687 8504	0.4424 6850	0.4176 5894	0.3942 6836
48	0.4893 6170	0.4612 8909	0.4348 5848	0.4099 7196	0.3865 3761
49	0.4821 2975	0.4539 1301	0.4273 7934	0.4024 2647	0.3789 5844
50	0.4750 0468	0.4466 5487	0.4200 2883	0.3950 1984	0.3715 2788

$$\frac{1}{(1+i)^n}$$

n	2⅛%	2¼%	2⅜%	2½%	2¾%
1	0.9791 9217	0.9779 9511	0.9768 0098	0.9756 0976	0.9732 3601
2	0.9588 1730	0.9564 7444	0.9541 4015	0.9518 1440	0.9471 8833
3	0.9388 6639	0.9354 2732	0.9320 0503	0.9285 9941	0.9218 3779
4	0.9193 3061	0.9148 4335	0.9103 8342	0.9059 5064	0.8971 6573
5	0.9002 0133	0.8947 1232	0.8892 6342	0.8838 5429	0.8731 5400
6	0.8814 7009	0.8750 2427	0.8686 3337	0.8622 9687	0.8497 8491
7	0.8631 2861	0.8557 6946	0.8484 8193	0.8412 6524	0.8270 4128
8	0.8451 6878	0.8369 3835	0.8287 9798	0.8207 4657	0.8049 0635
9	0.8275 8264	0.8185 2161	0.8095 7067	0.8007 2836	0.7833 6385
10	0.8103 6244	0.8005 1013	0.7907 8942	0.7811 9840	0.7623 9791
11	0.7935 0056	0.7828 9499	0.7724 4388	0.7621 4478	0.7419 9310
12	0.7769 8953	0.7656 6748	0.7545 2394	0.7435 5589	0.7221 3440
13	0.7608 2206	0.7488 1905	0.7370 1972	0.7254 2038	0.7028 0720
14	0.7449 9100	0.7323 4137	0.7199 2158	0.7077 2720	0.6839 9728
15	0.7294 8935	0.7162 2628	0.7032 2010	0.6904 6556	0.6656 9078
16	0.7143 1026	0.7004 6580	0.6869 0609	0.6736 2493	0.6478 7424
17	0.6994 4701	0.6850 5212	0.6709 7053	0.6571 9506	0.6305 3454
18	0.6848 9303	0.6699 7763	0.6554 0467	0.6411 6591	0.6136 5892
19	0.6706 4189	0.6552 3484	0.6401 9993	0.6255 2772	0.5972 3496
20	0.6566 8729	0.6408 1647	0.6253 4791	0.6102 7094	0.5812 5057
21	0.6430 2305	0.6267 1538	0.6108 4045	0.5953 8629	0.5656 9398
22	0.6296 4313	0.6129 2457	0.5966 6955	0.5808 6467	0.5505 5375
23	0.6165 4162	0.5994 3724	0.5828 2740	0.5666 9724	0.5358 1874
24	0.6037 1273	0.5862 4668	0.5693 0637	0.5528 7535	0.5214 7809
25	0.5911 5077	0.5733 4639	0.5560 9902	0.5393 9059	0.5075 2126
26	0.5788 5021	0.5607 2997	0.5431 9807	0.5262 3472	0.4939 3796
27	0.5668 0559	0.5483 9117	0.5305 9640	0.5133 9973	0.4807 1821
28	0.5550 1159	0.5363 2388	0.5182 8708	0.5008 7778	0.4678 5227
29	0.5434 6300	0.5245 2213	0.5062 6333	0.4886 6125	0.4553 3068
30	0.5321 5471	0.5129 8008	0.4945 1852	0.4767 4269	0.4431 4421
31	0.5210 8173	0.5016 9201	0.4830 4617	0.4651 1481	0.4312 8391
32	0.5102 3915	0.4906 5233	0.4718 3997	0.4537 7055	0.4197 4103
33	0.4996 2217	0.4798 5558	0.4608 9374	0.4427 0298	0.4085 0708
34	0.4892 2612	0.4692 9641	0.4502 0146	0.4319 0534	0.3975 7380
35	0.4790 4638	0.4589 6960	0.4397 5722	0.4213 7107	0.3869 3314
36	0.4690 7846	0.4488 7002	0.4295 5529	0.4110 9372	0.3765 7727
37	0.4593 1796	0.4389 9268	0.4195 9002	0.4010 6705	0.3664 9856
38	0.4497 6055	0.4293 3270	0.4098 5594	0.3912 8492	0.3566 8959
39	0.4404 0201	0.4198 8528	0.4003 4769	0.3817 4139	0.3471 4316
40	0.4312 3819	0.4106 4575	0.3910 6001	0.3724 3062	0.3378 5222
41	0.4222 6506	0.4016 0954	0.3819 8780	0.3633 4695	0.3288 0995
42	0.4134 7864	0.3927 7216	0.3731 2606	0.3544 8483	0.3200 0968
43	0.4048 7505	0.3841 2925	0.3644 6990	0.3458 3886	0.3114 4495
44	0.3964 5047	0.3756 7653	0.3560 1455	0.3374 0376	0.3031 0944
45	0.3882 0120	0.3674 0981	0.3477 5536	0.3291 7440	0.2949 9702
46	0.3801 2357	0.3593 2500	0.3396 8778	0.3211 4576	0.2871 0172
47	0.3722 1402	0.3514 1809	0.3318 0735	0.3133 1294	0.2794 1773
48	0.3644 6906	0.3436 8518	0.3241 0975	0.3056 7116	0.2719 3940
49	0.3568 8524	0.3361 2242	0.3165 9072	0.2982 1576	0.2646 6122
50	0.3494 5924	0.3287 2608	0.3092 4612	0.2909 4221	0.2575 7783

$$\frac{1}{(1+i)^n}$$

n	3%	3¼%	3½%	3¾%	4%
1	0.9708 7379	0.9685 2300	0.9661 8357	0.9638 5542	0.9615 3846
2	0.9425 9591	0.9380 3681	0.9335 1070	0.9290 1727	0.9245 5621
3	0.9151 4166	0.9085 1022	0.9019 4271	0.8954 3834	0.8889 9636
4	0.8884 8705	0.8799 1305	0.8714 4223	0.8630 7310	0.8548 0419
5	0.8626 0878	0.8522 1603	0.8419 7317	0.8318 7768	0.8219 2711
6	0.8374 8426	0.8253 9083	0.8135 0064	0.8018 0981	0.7903 1453
7	0.8130 9151	0.7994 1000	0.7859 9096	0.7728 2874	0.7599 1781
8	0.7894 0923	0.7742 4698	0.7594 1156	0.7448 9517	0.7306 9021
9	0.7664 1673	0.7498 7601	0.7337 3097	0.7179 7125	0.7025 8674
10	0.7440 9391	0.7262 7216	0.7089 1881	0.6920 2048	0.6755 6417
11	0.7224 2128	0.7034 1129	0.6849 4571	0.6670 0769	0.6495 8093
12	0.7013 7988	0.6812 7002	0.6617 8330	0.6428 9898	0.6245 9705
13	0.6809 5134	0.6598 2568	0.6394 0415	0.6196 6167	0.6005 7409
14	0.6611 1781	0.6390 5635	0.6177 8179	0.5972 6426	0.5774 7508
15	0.6418 6195	0.6189 4078	0.5968 9062	0.5756 7639	0.5552 6450
16	0.6231 6694	0.5994 5838	0.5767 0591	0.5548 6881	0.5339 0818
17	0.6050 1645	0.5805 8923	0.5572 0378	0.5348 1331	0.5133 7325
18	0.5873 9461	0.5623 1402	0.5383 6114	0.5154 8271	0.4936 2812
19	0.5702 8603	0.5446 1407	0.5201 5569	0.4968 5080	0.4746 4242
20	0.5536 7575	0.5274 7125	0.5025 6588	0.4788 9234	0.4563 8695
21	0.5375 4928	0.5108 6804	0.4855 7090	0.4615 8298	0.4388 3360
22	0.5218 9250	0.4947 8745	0.4691 5063	0.4448 9925	0.4219 5539
23	0.5066 9175	0.4792 1302	0.4532 8563	0.4288 1856	0.4057 2633
24	0.4919 3374	0.4641 2884	0.4379 5713	0.4133 1910	0.3901 2147
25	0.4776 0557	0.4495 1945	0.4231 4699	0.3983 7985	0.3751 1680
26	0.4636 9473	0.4353 6993	0.4088 3767	0.3839 8058	0.3606 8923
27	0.4501 8906	0.4216 6579	0.3950 1224	0.3701 0176	0.3468 1657
28	0.4370 7675	0.4083 9302	0.3816 5434	0.3567 2459	0.3334 7747
29	0.4243 4636	0.3955 3803	0.3687 4815	0.3438 3093	0.3206 5141
30	0.4119 8676	0.3830 8768	0.3562 7841	0.3314 0331	0.3083 1867
31	0.3999 8715	0.3710 2923	0.3442 3035	0.3194 2487	0.2964 6026
32	0.3883 3703	0.3593 5035	0.3325 8971	0.3078 7940	0.2850 5794
33	0.3770 2625	0.3480 3908	0.3213 4271	0.2967 5123	0.2740 9417
34	0.3660 4490	0.3370 8385	0.3104 7605	0.2860 2528	0.2635 5209
35	0.3553 8340	0.3264 7346	0.2999 7686	0.2756 8702	0.2534 1547
36	0.3450 3243	0.3161 9706	0.2898 3272	0.2657 2242	0.2436 6872
37	0.3349 8294	0.3062 4413	0.2800 3161	0.2561 1800	0.2342 9685
38	0.3252 2615	0.2966 0448	0.2705 6194	0.2468 6072	0.2252 8543
39	0.3157 5355	0.2872 6826	0.2614 1250	0.2379 3805	0.2166 2061
40	0.3065 5684	0.2782 2592	0.2525 7247	0.2293 3788	0.2082 8904
41	0.2976 2800	0.2694 6820	0.2440 3137	0.2210 4855	0.2002 7793
42	0.2889 5922	0.2609 8615	0.2357 7910	0.2130 5885	0.1925 7493
43	0.2805 4294	0.2527 7109	0.2278 0590	0.2053 5793	0.1851 6820
44	0.2723 7178	0.2448 1462	0.2201 0231	0.1979 3535	0.1780 4635
45	0.2644 3862	0.2371 0859	0.2126 5924	0.1907 8106	0.1711 9841
46	0.2567 3653	0.2296 4512	0.2054 6787	0.1838 8536	0.1646 1386
47	0.2492 5876	0.2224 1658	0.1985 1968	0.1772 3890	0.1582 8256
48	0.2419 9880	0.2154 1558	0.1918 0645	0.1708 3268	0.1521 9476
49	0.2349 5029	0.2086 3494	0.1853 2024	0.1646 5800	0.1463 4112
50	0.2281 0708	0.2020 6774	0.1790 5337	0.1587 0651	0.1407 1262

$$\frac{1}{(1+i)^n}$$

n	4¼%	4½%	4¾%	5%	5½%
1	0.9592 3261	0.9569 3780	0.9546 5394	0.9523 8095	0.9478 6730
2	0.9201 2721	0.9157 2995	0.9113 6414	0.9070 2948	0.8984 5242
3	0.8826 1603	0.8762 9660	0.8700 3737	0.8638 3760	0.8516 1366
4	0.8466 3408	0.8385 6134	0.8305 8460	0.8227 0247	0.8072 1674
5	0.8121 1902	0.8024 5105	0.7929 2086	0.7835 2617	0.7651 3435
6	0.7790 1105	0.7678 9574	0.7569 6502	0.7462 1540	0.7252 4583
7	0.7472 5281	0.7348 2846	0.7226 3964	0.7106 8133	0.6874 3681
8	0.7167 8926	0.7031 8513	0.6898 7077	0.6768 3936	0.6515 9887
9	0.6875 6764	0.6729 0443	0.6585 8785	0.6446 0892	0.6176 2926
10	0.6595 3730	0.6439 2768	0.6287 2349	0.6139 1325	0.5854 3058
11	0.6326 4969	0.6161 9874	0.6002 1335	0.5846 7929	0.5549 1050
12	0.6068 5822	0.5896 6386	0.5729 9604	0.5568 3742	0.5259 8152
13	0.5821 1819	0.5642 7164	0.5470 1293	0.5303 2135	0.4985 6068
14	0.5583 8676	0.5399 7286	0.5222 0804	0.5050 6795	0.4725 6937
15	0.5356 2279	0.5167 2044	0.4985 2797	0.4810 1710	0.4479 3305
16	0.5137 8685	0.4944 6932	0.4759 2169	0.4581 1152	0.4245 8109
17	0.4928 4110	0.4731 7639	0.4543 4051	0.4362 9669	0.4024 4653
18	0.4727 4926	0.4528 0037	0.4337 3796	0.4155 2065	0.3814 6590
19	0.4534 7650	0.4333 0179	0.4140 6965	0.3957 3396	0.3615 7906
20	0.4349 8945	0.4146 4286	0.3952 9322	0.3768 8948	0.3427 2896
21	0.4172 5607	0.3967 8743	0.3773 6823	0.3589 4236	0.3248 6158
22	0.4002 4563	0.3797 0089	0.3602 5607	0.3418 4987	0.3079 2567
23	0.3839 2866	0.3633 5013	0.3439 1987	0.3255 7131	0.2918 7267
24	0.3682 7689	0.3477 0347	0.3283 2446	0.3100 6791	0.2766 5656
25	0.3532 6321	0.3327 3060	0.3134 3624	0.2953 0277	0.2622 3370
26	0.3388 6159	0.3184 0248	0.2992 2314	0.2812 4073	0.2485 6275
27	0.3250 4709	0.3046 9137	0.2856 5455	0.2678 4832	0.2356 0450
28	0.3117 9577	0.2915 7069	0.2727 0124	0.2550 9364	0.2233 2181
29	0.2990 8467	0.2790 1502	0.2603 3531	0.2429 4632	0.2116 7944
30	0.2868 9177	0.2670 0002	0.2485 3013	0.2313 7745	0.2006 4402
31	0.2751 9594	0.2555 0241	0.2372 6027	0.2203 5947	0.1901 8390
32	0.2639 7692	0.2444 9991	0.2265 0145	0.2098 6617	0.1802 6910
33	0.2532 1527	0.2339 7121	0.2162 3050	0.1998 7254	0.1708 7119
34	0.2428 9235	0.2238 9589	0.2064 2530	0.1903 5480	0.1619 6321
35	0.2329 9026	0.2142 5444	0.1970 6473	0.1812 9029	0.1535 1963
36	0.2234 9186	0.2050 2817	0.1881 2862	0.1726 5741	0.1455 1624
37	0.2143 8068	0.1961 9921	0.1795 9772	0.1644 3563	0.1379 3008
38	0.2056 4094	0.1877 5044	0.1714 5367	0.1566 0536	0.1307 3941
39	0.1972 5750	0.1796 6549	0.1636 7893	0.1491 4797	0.1239 2362
40	0.1892 1582	0.1719 2870	0.1562 5673	0.1420 4568	0.1174 6314
41	0.1815 0199	0.1645 2507	0.1491 7110	0.1352 8160	0.1113 3947
42	0.1741 0263	0.1574 4026	0.1424 0678	0.1288 3962	0.1055 3504
43	0.1670 0492	0.1506 6054	0.1359 4919	0.1227 0440	0.1000 3322
44	0.1601 9657	0.1441 7276	0.1297 8443	0.1168 6133	0.0948 1822
45	0.1536 6577	0.1379 6437	0.1238 9922	0.1112 9651	0.0898 7509
46	0.1474 0122	0.1320 2332	0.1182 8088	0.1059 9668	0.0851 8965
47	0.1413 9206	0.1263 3810	0.1129 1731	0.1009 4921	0.0807 4849
48	0.1356 2787	0.1208 9771	0.1077 9695	0.0961 4211	0.0765 3885
49	0.1300 9868	0.1156 9158	0.1029 0878	0.0915 6391	0.0725 4867
50	0.1247 9489	0.1107 0965	0.0982 4228	0.0872 0373	0.0687 6652

$$\frac{1}{(1+i)^n}$$

n	6%	6½%	7%	8%	9%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9259 2593	0.9174 3119
2	0.8899 9644	0.8816 5928	0.8734 3873	0.8573 3882	0.8416 7999
3	0.8396 1928	0.8278 4909	0.8162 9788	0.7938 3224	0.7721 8348
4	0.7920 9366	0.7773 2309	0.7628 9521	0.7350 2985	0.7084 2521
5	0.7472 5817	0.7298 8084	0.7129 8618	0.6805 8320	0.6499 3139
6	0.7049 6054	0.6853 3412	0.6663 4222	0.6301 6963	0.5962 6733
7	0.6650 5711	0.6435 0621	0.6227 4974	0.5834 9040	0.5470 3424
8	0.6274 1237	0.6042 3119	0.5820 0910	0.5402 6888	0.5018 6628
9	0.5918 9846	0.5673 5323	0.5439 3374	0.5002 4897	0.4604 2778
10	0.5583 9478	0.5327 2604	0.5083 4929	0.4631 9349	0.4224 1081
11	0.5267 8753	0.5002 1224	0.4750 9280	0.4288 8286	0.3875 3285
12	0.4969 6936	0.4696 8285	0.4440 1196	0.3971 1376	0.3555 3473
13	0.4688 3902	0.4410 1676	0.4149 6445	0.3676 9792	0.3261 7865
14	0.4423 0096	0.4141 0025	0.3878 1724	0.3404 6104	0.2992 4647
15	0.4172 6506	0.3888 2652	0.3624 4602	0.3152 4170	0.2745 3804
16	0.3936 4628	0.3650 9533	0.3387 3460	0.2918 9047	0.2518 6976
17	0.3713 6442	0.3428 1251	0.3165 7439	0.2702 6895	0.2310 7318
18	0.3503 4379	0.3218 8969	0.2958 6392	0.2502 4903	0.2119 9374
19	0.3305 1301	0.3022 4384	0.2765 0832	0.2317 1206	0.1944 8967
20	0.3118 0473	0.2837 9703	0.2584 1900	0.2145 4821	0.1784 3089
21	0.2941 5540	0.2664 7608	0.2415 1309	0.1986 5575	0.1636 9806
22	0.2775 0510	0.2502 1228	0.2257 1317	0.1839 4051	0.1501 8171
23	0.2617 9726	0.2349 4111	0.2109 4688	0.1703 1528	0.1377 8139
24	0.2469 7855	0.2206 0198	0.1971 4662	0.1576 9934	0.1264 0494
25	0.2329 9863	0.2071 3801	0.1842 4918	0.1460 1790	0.1159 6784
26	0.2198 1003	0.1944 9579	0.1721 9549	0.1352 0176	0.1063 9251
27	0.2073 6795	0.1826 2515	0.1609 3037	0.1251 8682	0.0976 0781
28	0.1956 3014	0.1714 7902	0.1504 0221	0.1159 1372	0.0895 4845
29	0.1845 5674	0.1610 1316	0.1405 6282	0.1073 2752	0.0821 5454
30	0.1741 1013	0.1511 8607	0.1313 6712	0.0993 7733	0.0753 7114
31	0.1642 5484	0.1419 5875	0.1227 7301	0.0920 1605	0.0691 4783
32	0.1549 5740	0.1332 9460	0.1147 4113	0.0852 0005	0.0634 3838
33	0.1461 8622	0.1251 5925	0.1072 3470	0.0788 8893	0.0582 0035
34	0.1379 1153	0.1175 2042	0.1002 1934	0.0730 4531	0.0533 9481
35	0.1301 0522	0.1103 4781	0.0936 6294	0.0676 3454	0.0489 8607
36	0.1227 4077	0.1036 1297	0.0875 3546	0.0626 2458	0.0449 4135
37	0.1157 9318	0.0972 8917	0.0818 0884	0.0579 8572	0.0412 3059
38	0.1092 3885	0.0913 5134	0.0764 5686	0.0536 9048	0.0378 2623
39	0.1030 5552	0.0857 7590	0.0714 5501	0.0497 1341	0.0347 0296
40	0.0972 2219	0.0805 4075	0.0667 8038	0.0460 3093	0.0318 3758
41	0.0917 1905	0.0756 2512	0.0624 1157	0.0426 2123	0.0292 0879
42	0.0865 2740	0.0710 0950	0.0583 2857	0.0394 6411	0.0267 9706
43	0.0816 2962	0.0666 7559	0.0545 1268	0.0365 4084	0.0245 8446
44	0.0770 0908	0.0626 0619	0.0509 4643	0.0338 3411	0.0225 5455
45	0.0726 5007	0.0587 8515	0.0476 1349	0.0313 2788	0.0206 9224
46	0.0685 3781	0.0551 9733	0.0444 9859	0.0290 0730	0.0189 8371
47	0.0646 5831	0.0518 2848	0.0415 8747	0.0268 5861	0.0174 1625
48	0.0609 9840	0.0486 6524	0.0388 6679	0.0248 6908	0.0159 7821
49	0.0575 4566	0.0456 9506	0.0363 2410	0.0230 2693	0.0146 5891
50	0.0542 8836	0.0429 0616	0.0339 4776	0.0213 2123	0.0134 4854

$$\frac{(1+i)^n - 1}{i}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.005 0000	2.010 0000	2.012 5000	2.015 0000	2.017 5000	2.020 0000
3	3.015 0250	3.030 1000	3.037 6562	3.045 2250	3.052 8063	3.060 4000
4	4.030 1001	4.060 4010	4.075 6270	4.090 9034	4.106 2304	4.121 6080
5	5.050 2506	5.101 0050	5.126 5723	5.152 2669	5.178 0894	5.204 0402
6	6.075 5019	6.152 0151	6.190 6544	6.229 5509	6.268 7060	6.308 1210
7	7.105 8794	7.213 5352	7.268 0376	7.322 9942	7.378 4083	7.434 2834
8	8.141 4088	8.285 6706	8.358 8881	8.432 8391	8.507 5305	8.582 9691
9	9.182 1158	9.368 5273	9.463 3742	9.559 3317	9.656 4122	9.754 6284
10	10.228 0264	10.462 2125	10.581 6664	10.702 7217	10.825 3995	10.949 7210
11	11.279 1665	11.566 8347	11.713 9372	11.863 2625	12.014 8439	12.168 7154
12	12.335 5624	12.682 5030	12.860 3614	13.041 2114	13.225 1037	13.412 0897
13	13.397 2402	13.809 3280	14.021 1159	14.236 8296	14.456 5430	14.680 3315
14	14.464 2264	14.947 4213	15.196 3799	15.450 3821	15.709 5325	15.973 9382
15	15.536 5475	16.096 8955	16.386 3346	16.682 1378	16.984 4494	17.293 4169
16	16.614 2303	17.257 8645	17.591 1638	17.932 3698	18.281 6772	18.639 2853
17	17.697 3014	18.430 4431	18.811 0534	19.201 3554	19.601 6066	20.012 0710
18	18.785 7879	19.614 7476	20.046 1915	20.489 3757	20.944 6347	21.412 3124
19	19.879 7168	20.810 8950	21.296 7689	21.796 7164	22.311 1658	22.840 5586
20	20.979 1154	22.019 0040	22.562 9785	23.123 6671	23.701 6112	24.297 3698
21	22.084 0110	23.239 1940	23.845 0158	24.470 5221	25.116 3894	25.783 3172
22	23.194 4311	24.471 5860	25.143 0785	25.837 5799	26.555 9262	27.298 9835
23	24.310 4032	25.716 3018	26.457 3670	27.225 1436	28.020 6549	28.844 9632
24	25.431 9552	26.973 4648	27.788 0840	28.633 5208	29.511 0164	30.421 8625
25	26.559 1150	28.243 1995	29.135 4351	30.063 0236	31.027 4592	32.030 2997
26	27.691 9106	29.525 6315	30.499 6280	31.513 9690	32.570 4397	33.670 9057
27	28.830 3702	30.820 8878	31.880 8734	32.986 6785	34.140 4224	35.344 3238
28	29.974 5220	32.129 0967	33.279 3843	34.481 4787	35.737 8798	37.051 2103
29	31.124 3946	33.450 3877	34.695 3766	35.998 7009	37.363 2927	38.792 2345
30	32.280 0166	34.784 8915	36.129 0688	37.538 6814	39.017 1503	40.568 0792
31	33.441 4167	36.132 7404	37.580 6822	39.101 7616	40.699 9504	42.379 4408
32	34.608 6238	37.494 0678	39.050 4407	40.688 2880	42.412 1996	44.227 0296
33	35.781 6669	38.869 0085	40.538 5712	42.298 6123	44.154 4131	46.111 5702
34	36.960 5752	40.257 6986	42.045 3033	43.933 0915	45.927 1153	48.033 8016
35	38.145 3781	41.660 2756	43.570 8696	45.592 0879	47.730 8398	49.994 4776
36	39.336 1050	43.076 8784	45.115 5055	47.275 9692	49.566 1295	51.994 3672
37	40.532 7855	44.507 6471	46.679 4493	48.985 1087	51.433 5368	54.034 2545
38	41.735 4494	45.952 7236	48.262 9424	50.719 8854	53.333 6236	56.114 9396
39	42.944 1267	47.412 2508	49.866 2292	52.480 6837	55.266 9621	58.237 2384
40	44.158 8473	48.886 3734	51.489 5571	54.267 8939	57.234 1339	60.401 9832
41	45.379 6415	50.375 2371	53.133 1765	56.081 9123	59.235 7312	62.610 0228
42	46.606 5397	51.878 9895	54.797 3412	57.923 1410	61.272 3565	64.862 2233
43	47.839 5724	53.397 7794	56.482 3080	59.791 9881	63.344 6228	67.159 4678
44	49.078 7703	54.931 7572	58.188 3369	61.688 8679	65.453 1537	69.502 6571
45	50.324 1642	56.481 0747	59.915 6911	63.614 2010	67.598 5839	71.892 7103
46	51.575 7850	58.045 8855	61.664 6372	65.568 4140	69.781 5591	74.330 5645
47	52.833 6639	59.626 3443	63.435 4452	67.551 9402	72.002 7364	76.817 1758
48	54.097 8322	61.222 6078	65.228 3882	69.565 2193	74.262 7843	79.353 5193
49	55.368 3214	62.834 8338	67.043 7431	71.608 6976	76.562 3830	81.940 5897
50	56.645 1630	64.463 1822	68.881 7899	73.682 8280	78.902 2247	84.579 4015

$$\frac{(1+i)^n-1}{i}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.022 5000	2.025 0000	2.027 5000	2.030 0000	2.035 0000	2.040 0000
3	3.068 0063	3.075 6250	3.083 2563	3.090 9000	3.106 2250	3.121 6000
4	4.137 0364	4.152 5156	4.168 0458	4.183 6270	4.214 9429	4.246 4640
5	5.230 1197	5.256 3285	5.282 6671	5.309 1358	5.362 4659	5.416 3226
6	6.347 7974	6.387 7367	6.427 9404	6.468 4099	6.550 1522	6.632 9755
7	7.490 6228	7.547 4302	7.604 7088	7.662 4622	7.779 4075	7.898 2945
8	8.659 1619	8.736 1159	8.813 8383	8.892 3361	9.051 6868	9.214 2263
9	9.853 9930	9.954 5188	10.056 2188	10.159 1061	10.368 4958	10.582 7953
10	11.075 7078	11.203 3818	11.332 7648	11.463 8793	11.731 3932	12.006 1071
11	12.324 9113	12.483 4663	12.644 4159	12.807 7957	13.141 9919	13.486 3514
12	13.602 2218	13.795 5530	13.992 1373	14.192 0296	14.601 9616	15.025 8055
13	14.908 2718	15.140 4418	15.376 9211	15.617 7905	16.113 0303	16.626 8377
14	16.243 7079	16.518 9528	16.799 7864	17.086 3242	17.676 9864	18.291 9112
15	17.609 1913	17.931 9267	18.261 7805	18.598 9139	19.295 6809	20.023 5876
16	19.005 3981	19.380 2248	19.763 9795	20.156 8813	20.971 0297	21.824 5311
17	20.433 0196	20.864 7305	21.307 4889	21.761 5877	22.705 0158	23.697 5124
18	21.892 7625	22.386 3487	22.893 4449	23.414 4354	24.499 6913	25.645 4129
19	23.385 3497	23.946 0074	24.523 0146	25.116 8684	26.357 1805	27.671 2294
20	24.911 5200	25.544 6576	26.197 3975	26.870 3745	28.279 6818	29.778 0786
21	26.472 0292	27.183 2741	27.917 8259	28.676 4857	30.269 4707	31.969 2017
22	28.067 6499	28.862 8559	29.685 5662	30.536 7803	32.328 9022	34.247 9698
23	29.699 1720	30.584 4273	31.501 9192	32.452 8837	34.460 4137	36.617 8886
24	31.367 4034	32.349 0380	33.368 2220	34.426 4702	36.666 5282	39.082 6041
25	33.073 1700	34.157 7639	35.285 8481	36.459 2643	38.949 8567	41.645 9083
26	34.817 3163	36.011 7080	37.256 2089	38.553 0423	41.313 1017	44.311 7446
27	36.600 7059	37.912 0007	39.280 7547	40.709 6335	43.759 0602	47.084 2144
28	38.424 2218	39.859 8008	41.360 9754	42 930 9225	46.290 6273	49.967 5830
29	40.288 7668	41.856 2958	43.498 4022	45.218 8502	48.910 7993	52.966 2863
30	42.195 2640	43.902 7032	45.694 6083	47.575 4157	51.622 6773	56.084 9378
31	44.144 6575	46.000 2707	47.951 2100	50.002 6782	54.429 4710	59.328 3353
32	46.137 9123	48.150 2775	50.269 8683	52.502 7585	57.334 5025	62.701 4687
33	48.176 0153	50.354 0345	52.652 2807	55.077 8413	60.341 2101	66.209 5274
34	50.259 9756	52.612 8853	55.100 2277	57.730 1765	63.453 1524	69.857 9085
35	52.390 8251	54.928 2074	57.615 4839	60.462 0818	66.674 0127	73.652 2249
36	54.569 6186	57.301 4126	60.199 9097	63.275 9443	70.007 6032	77.598 3139
37	56.797 4351	59.733 9479	62.855 4072	66.174 2226	73.457 8693	81.702 2464
38	59.075 3774	62.227 2966	65.583 9309	69.159 4493	77.028 8947	85.970 3363
39	61.404 5783	64.782 9791	68.387 4890	72.234 2328	80.724 9060	90.409 1497
40	63.786 1762	67.402 5535	71.268 1450	75.401 2597	84.550 2778	95.025 5157
41	66.221 3652	70.087 6174	74.228 0190	78.663 2975	88.509 5375	99.826 5363
42	68.711 3459	72.839 8078	77.269 2895	82.023 1965	92.607 3713	104.819 5978
43	71.257 3512	75.660 8030	80.394 1950	85.483 8923	96.848 6293	110.012 3817
44	73.860 6416	78.552 3231	83.605 0353	89.048 4091	101.238 3313	115.412 8770
45	76.522 5061	81.516 1312	86.904 1738	92.719 8614	105.781 6729	121.029 3920
46	79.244 2624	84.554 0344	90.294 0386	96.501 4572	110.484 0315	126.870 5677
47	82.027 2583	87.667 8853	93.777 1246	100.396 5010	115.350 9726	132.945 3904
48	84.872 8717	90.859 5824	97.355 9956	104.408 3960	120.388 2566	139.263 2060
49	87.782 5113	94.131 0720	101.033 2854	108.540 6479	125.601 8456	145.833 7343
50	90.757 6178	97.484 3488	104.811 7008	112.796 8673	130.997 9102	152.667 0837

$$\frac{(1+i)^n - 1}{i}$$

n	4½%	5%	5½%	6%	7%	8%
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.045 0000	2.050 0000	2.055 0000	2.060 0000	2.070 0000	2.080 0000
3	3.137 0250	3.152 5000	3.168 0250	3.183 6000	3.214 9000	3.246 4000
4	4.278 1911	4.310 1250	4.342 2664	4.374 6160	4.439 9430	4.506 1120
5	5.470 7097	5.525 6313	5.581 0910	5.637 0930	5.750 7390	5.866 6010
6	6.716 8917	6.801 9128	6.888 0510	6.975 3185	7.153 2907	7.335 9290
7	8.019 1518	8.142 0085	8.266 8938	8.393 8377	8.654 0211	8.922 8034
8	9.380 0136	9.549 1089	9.721 5730	9.897 4679	10.259 8026	10.636 6276
9	10.802 1142	11.026 5643	11.256 2595	11.491 3160	11.977 9888	12.487 5578
10	12.288 2094	12.577 8925	12.875 3538	13.180 7949	13.816 4480	14.486 5625
11	13.841 1788	14.206 7872	14.583 4983	14.971 6426	15.783 5993	16.645 4875
12	15.464 0318	15.917 1265	16.385 5907	16.869 9412	17.888 4513	18.977 1265
13	17.159 9133	17.712 9829	18.286 7981	18.882 1377	20.140 6429	21.495 2966
14	18.932 1094	19.598 6320	20.292 5720	21.015 0659	22.550 4879	24.214 9203
15	20.784 0543	21.578 5636	22.408 6635	23.275 9699	25.129 0220	27.152 1139
16	22.719 3367	23.657 4918	24.641 1400	25.672 5281	27.888 0536	30.324 2830
17	24.741 7069	25.840 3664	26.996 4027	28.212 8798	30.840 2173	33.750 2257
18	26.855 0837	28.132 3847	29.481 2048	30.905 6526	33.999 0325	37.450 2437
19	29.063 5625	30.539 0039	32.102 6711	33.759 9917	37.378 9648	41.446 2632
20	31.371 4228	33.065 9541	34.868 3180	36.785 5912	40.995 4923	45.761 9643
21	33.783 1368	35.719 2518	37.786 0755	39.992 7267	44.865 1768	50.422 9214
22	36.303 3780	38.505 2144	40.864 3097	43.392 2903	49.005 7392	55.456 7552
23	38.937 0300	41.430 4751	44.111 8467	46.995 8277	53.436 1409	60.893 2956
24	41.689 1963	44.501 9989	47.537 9983	50.815 5774	58.176 6708	66.764 7592
25	44.565 2102	47.727 0988	51.152 5882	54.864 5120	63.249 0377	73.105 9400
26	47.570 6446	51.113 4538	54.965 9805	59.156 3827	68.676 4704	79.954 4152
27	50.711 3236	54.669 1265	58.989 1094	63.705 7657	74.483 8233	87.350 7684
28	53.993 3332	58.402 5828	63.233 5105	68.528 1116	80.697 6909	95.338 8298
29	57.423 0332	62.322 7119	67.711 3535	73.639 7983	87.346 5293	103.965 9362
30	61.007 0697	66.438 8475	72.435 4780	79.058 1862	94.460 7863	113.283 2111
31	64.752 3878	70.760 7899	77.419 4293	84.801 6774	102.073 0414	123.345 8680
32	68.666 2452	75.298 8294	82.677 4979	90.889 7780	110.218 1543	134.213 5374
33	72.756 2263	80.063 7708	88.224 7603	97.343 1647	118.933 4251	145.950 6204
34	77.030 2565	85.066 9594	94.077 1221	104.183 7546	128.258 7648	158.626 6701
35	81.496 6180	90.320 3074	100.251 3638	111.434 7799	138.236 8784	172.316 8037
36	86.163 9658	95.836 3227	106.765 1888	119.120 8667	148.913 4598	187.102 1480
37	91.041 3443	101.628 1389	113.637 2742	127.268 1187	160.337 4020	203.070 3198
38	96.138 2048	107.709 5458	120.887 3243	135.904 2058	172.561 0202	220.315 9454
39	101.464 4240	114.095 0231	128.536 1271	145.058 4581	185.640 2916	238.941 2210
40	107.030 3231	120.799 7742	136.605 6141	154.761 9656	199.635 1120	259.056 5187
41	112.846 6876	127.839 7630	145.118 9229	165.047 6836	214.609 5698	280.781 0402
42	118.924 7885	135.231 7511	154.100 4636	175.950 5446	230.632 2397	304.243 5234
43	125.276 4040	142.993 3387	163.575 9891	187.507 5772	247.776 4695	329.583 0053
44	131.913 8422	151.143 0056	173.572 6685	199.758 0319	266.120 8513	356.949 6457
45	138.849 9651	159.700 1559	184.119 1653	212.743 5138	285.749 3108	386.505 6174
46	146.098 2135	168.685 1637	195.245 7194	226.508 1246	306.751 7626	418.426 0668
47	153.672 6331	178.119 4219	206.984 2339	241.098 6121	329.224 3860	452.900 1521
48	161.587 9016	188.025 3929	219.368 3668	256.564 5288	353.270 0930	490.132 1643
49	169.859 3572	198.426 6626	232.433 6270	272.958 4006	378.998 9995	530.342 7374
50	178.503 0283	209.347 9957	246.217 4765	290.335 9046	406.528 9295	573.770 1564

$$\frac{1-\frac{1}{(1+i)^n}}{i}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	0.982 8010	0.980 3922
2	1.985 0994	1.970 3951	1.963 1154	1.955 8834	1.948 6988	1.941 5609
3	2.970 2481	2.940 0852	2.926 5337	2.912 2004	2.897 9840	2.883 8833
4	3.950 4957	3.901 9656	3.878 0580	3.854 3847	3.830 9425	3.807 7287
5	4.925 8663	4.853 4312	4.817 8350	4.782 6450	4.747 8551	4.713 4595
6	5.896 3844	5.795 4765	5.746 0099	5.697 1872	5.648 9976	5.601 4309
7	6.862 0740	6.728 1945	6.662 7258	6.598 2140	6.534 6414	6.471 9911
8	7.822 9592	7.651 6778	7.568 1243	7.485 9251	7.405 0530	7.325 4814
9	8.779 0639	8.566 0176	8.462 3450	8.360 5173	8.260 4943	8.162 2367
10	9.730 4119	9.471 3045	9.345 5259	9.222 1846	9.101 2229	8.982 5850
11	10.677 0267	10.367 6282	10.217 8034	10.071 1178	9.927 4918	9.786 8481
12	11.618 9321	11.255 0775	11.079 3120	10.907 5052	10.739 5497	10.575 3412
13	12.556 1513	12.133 7401	11.930 1847	11.731 5322	11.537 6410	11.348 3738
14	13.488 7078	13.003 7030	12.770 5528	12.543 3815	12.322 0059	12.106 2488
15	14.416 6246	13.865 0525	13.600 5459	13.343 2330	13.092 8805	12.849 2635
16	15.339 9250	14.717 8738	14.420 2923	14.131 2641	13.850 4968	13.577 7093
17	16.258 6319	15.562 2513	15.229 9183	14.907 6493	14.595 0828	14.291 8719
18	17.172 7680	16.398 2686	16.029 5489	15.672 5609	15.326 8627	14.992 0313
19	18.082 3562	17.226 0085	16.819 3076	16.426 1684	16.046 0567	15.678 4620
20	18.987 4192	18.045 5530	17.599 3161	17.168 6388	16.752 8813	16.351 4333
21	19.887 9792	18.856 9831	18.369 6950	17.900 1367	17.447 5492	17.011 2092
22	20.784 0590	19.660 3793	19.130 5629	18.620 8244	18.130 2695	17.658 0482
23	21.675 6806	20.455 8211	19.882 0374	19.330 8615	18.801 2476	18.292 2041
24	22.562 8662	21.243 3873	20.624 2345	20.030 4054	19.460 6857	18.913 9256
25	23.445 6380	22.023 1557	21.357 2686	20.719 6112	20.108 7820	19.523 4565
26	24.324 0179	22.795 2037	22.081 2530	21.398 6317	20.745 7317	20.121 0358
27	25.198 0278	23.559 6076	22.796 2992	22.067 6175	21.371 7264	20.706 8978
28	26.067 6894	24.316 4432	23.502 5178	22.726 7167	21.986 9547	21.281 2724
29	26.933 0242	25.065 7853	24.200 0176	23.376 0756	22.591 6017	21.844 3847
30	27.794 0540	25.807 7082	24.888 9062	24.015 8380	23.185 8493	22.396 4556
31	28.650 8000	26.542 2854	25.569 2901	24.646 1458	23.769 8765	22.937 7015
32	29.503 2836	27.269 5895	26.241 2742	25.267 1387	24.343 8590	23.468 3348
33	30.351 5259	27.989 6926	26.904 9622	25.878 9544	24.907 9695	23.988 5636
34	31.195 5482	28.702 6659	27.560 4564	26.481 7285	25.462 3779	24.498 5917
35	32.035 3713	29.408 5801	28.207 8582	27.075 5946	26.007 2510	24.998 6193
36	32.871 0162	30.107 5050	28.847 2674	27.660 6843	26.542 7528	25.488 8425
37	33.702 5037	30.799 5099	29.478 7826	28.237 1274	27.069 0446	25.969 4534
38	34.529 8544	31.484 6633	30.102 5013	28.805 0516	27.586 2846	26.440 6406
39	35.353 0890	32.163 0330	30.718 5198	29.364 5829	28.094 6286	26.902 5888
40	36.172 2279	32.834 6861	31.326 9332	29.915 8452	28.594 2296	27.355 4792
41	36.987 2914	33.499 6892	31.927 8352	30.458 9608	29.085 2379	27.799 4895
42	37.798 2999	34.158 1081	32.521 3187	30.994 0500	29.567 8014	28.234 7936
43	38.605 2735	34.810 0081	33.107 4753	31.521 2316	30.042 0652	28.661 5623
44	39.408 2324	35.455 4535	33.686 3954	32.040 6222	30.508 1722	29.079 9631
45	40.207 1964	36.094 5084	34.258 1682	32.552 3372	30.966 2626	29.490 1599
46	41.002 1855	36.727 2361	34.822 8822	33.056 4898	31.416 4743	29.892 3136
47	41.793 2194	37.353 6991	35.380 6244	33.553 1920	31.858 9428	30.286 5820
48	42.580 3178	37.973 9595	35.931 4809	34.042 5537	32.293 8013	30.673 1196
49	43.363 5003	38.588 0787	36.475 5367	34.524 6834	32.721 1806	31.052 0780
50	44.142 7864	39.196 1175	37.012 8757	34.999 6881	33.141 2095	31.423 6059

$$\frac{1-\frac{1}{(1+i)^n}}{i}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	0.977 9951	0.975 6098	0.973 2360	0.970 8738	0.966 1836	0.961 5385
2	1.934 4696	1.927 4242	1.920 4243	1.913 4697	1.899 6943	1.886 0947
3	2.869 8969	2.856 0236	2.842 2621	2.828 6114	2.801 6370	2.775 0910
4	3.784 7402	3.761 9742	3.739 4279	3.717 0984	3.673 0792	3.629 8952
5	4.679 4525	4.645 8285	4.612 5819	4.579 7072	4.515 0524	4.451 8223
6	5.554 4768	5.508 1254	5.462 3668	5.417 1914	5.328 5530	5.242 1369
7	6.410 2463	6.349 3906	6.289 4081	6.230 2830	6.114 5440	6.002 0547
8	7.247 1846	7.170 1372	7.094 3144	7.019 6922	6.873 9555	6.732 7449
9	8.065 7062	7.970 8655	7.877 6783	7.786 1089	7.607 6865	7.435 3316
10	8.866 2164	8.752 0639	8.640 0762	8.530 2028	8.316 6053	8.110 3958
11	9.649 1113	9.514 2087	9.382 0693	9.252 6241	9.001 5510	8.760 4767
12	10.414 7788	10.257 7646	10.104 2037	9.954 0040	9.663 3343	9.385 0738
13	11.163 5979	10.983 1850	10.807 0109	10.634 9553	10.302 7385	9.985 6479
14	11.895 9392	11.690 0122	11.491 0081	11.296 0731	10.920 5203	10.563 1229
15	12.612 1655	12.381 3777	12.156 6989	11.937 9351	11.517 4109	11.118 3874
16	13.312 6313	13.055 0027	12.804 5732	12.561 1020	12.094 1168	11.652 2956
17	13.997 6834	13.712 1977	13.435 1077	13.166 1185	12.651 3206	12.165 6689
18	14.667 6611	14.353 3636	14.048 7666	13.753 5131	13.189 6817	12.659 2970
19	15.322 8959	14.978 8913	14.646 0016	14.323 7991	13.709 8374	13.133 9394
20	15.963 7124	15.589 1623	15.227 2521	14.877 4749	14.212 4033	13.590 3263
21	16.590 4278	16.184 5486	15.792 9461	15.415 0241	14.697 9742	14.029 1600
22	17.203 3523	16.765 4132	16.343 4999	15.936 9166	15.167 1248	14.451 1153
23	17.802 7896	17.332 1105	16.879 3186	16.443 6084	15.620 4105	14.856 8417
24	18.389 0362	17.884 9858	17.400 7967	16.935 5421	16.058 3676	15.246 9631
25	18.962 3826	18.424 3764	17.908 3180	17.413 1477	16.481 5146	15.622 0799
26	19.523 1126	18.950 6111	18.402 2559	17.876 8424	16.890 3523	15.982 7692
27	20.071 5038	19.464 0109	18.882 9741	18.327 0315	17.285 3645	16.329 5858
28	20.607 8276	19.964 8887	19.350 8264	18.764 1082	17.667 0189	16.663 0632
29	21.132 3498	20.453 5499	19.806 1571	19.188 4546	18.035 7670	16.983 7146
30	21.645 3299	20.930 2926	20.249 3013	19.600 4414	18.392 0454	17.292 0333
31	22.147 0219	21.395 4074	20.680 5852	20.000 4285	18.736 2758	17.588 4936
32	22.637 6742	21.849 1780	21.100 3262	20.388 7655	19.068 8655	17.873 5515
33	23.117 5298	22.291 8809	21.508 8333	20.765 7918	19.390 2082	18.147 6457
34	23.586 8262	22.723 7863	21.906 4071	21.131 8367	19.700 6842	18.411 1978
35	24.045 7958	23.145 1573	22.293 3403	21.487 2201	20.000 6611	18.664 6132
36	24.494 6658	23.556 2511	22.669 9175	21.832 2525	20.290 4938	18.908 2820
37	24.933 6585	23.957 3181	23.036 4161	22.167 2354	20.570 5254	19.142 5788
38	25.362 9912	24.348 6030	23.393 1057	22.492 4616	20.841 0874	19.367 8642
39	25.782 8765	24.730 3444	23.740 2488	22.808 2151	21.102 4999	19.584 4848
40	26.193 5222	25.102 7751	24.078 1011	23.114 7720	21.355 0723	19.792 7739
41	26.595 1317	25.466 1220	24.406 9110	23.412 4000	21.599 1037	19.993 0518
42	26.987 9039	25.820 6068	24.726 9207	23.701 3592	21.834 8828	20.185 6267
43	27.372 0332	26.166 4457	25.038 3656	23.981 9021	22.062 6887	20.370 7949
44	27.747 7097	26.503 8495	25.341 4751	24.254 2739	22.282 7910	20.548 8413
45	28.115 1195	26.833 0239	25.636 4721	24.518 7125	22.495 4503	20.720 0397
46	28.474 4445	27.154 1696	25.923 5738	24.775 4491	22.700 9181	20.884 6536
47	28.825 8626	27.467 4826	26.202 9915	25.024 7078	22.899 4378	21.042 9361
48	29.169 5478	27.773 1537	26.474 9309	25.266 7066	23.091 2443	21.195 1309
49	29.505 6702	28.071 3695	26.739 5922	25.501 6569	23.276 5645	21.341 4720
50	29.834 3963	28.362 3117	26.997 1700	25.729 7640	23.455 6179	21.482 1846

$$\frac{1-\frac{1}{(1+i)^n}}{i}$$

n	4½%	5%	5½%	6%	7%	8%
1	0.956 9378	0.952 3810	0.947 8673	0.943 3962	0.934 5794	0.925 9259
2	1.872 6678	1.859 4104	1.846 3197	1.833 3927	1.808 0182	1.783 2648
3	2.748 9644	2.723 2480	2.697 9334	2.673 0120	2.624 3160	2.577 0970
4	3.587 5257	3.545 9505	3.505 1501	3.465 1056	3.387 2113	3.312 1268
5	4.389 9767	4.329 4767	4.270 2845	4.212 3638	4.100 1974	3.992 7100
6	5.157 8725	5.075 6923	4.995 5303	4.917 3243	4.766 5397	4.622 8797
7	5.892 7009	5.786 3731	5.682 9671	5.582 3814	5.389 2894	5.206 3701
8	6.595 8861	6.463 2128	6.334 5660	6.209 7938	5.971 2985	5.746 6389
9	7.268 7905	7.107 8217	6.952 1953	6.801 6923	6.515 2323	6.246 8879
10	7.912 7182	7.721 7349	7.537 6258	7.360 0871	7.023 5816	6.710 0814
11	8.528 9169	8.306 4142	8.092 5363	7.886 8746	7.498 6744	7.138 9643
12	9.118 5808	8.863 2516	8.618 5179	8.383 8439	7.942 6863	7.536 0780
13	9.682 8524	9.393 5730	9.117 0785	8.852 6830	8.357 6508	7.903 7759
14	10.222 8253	9.898 6409	9.589 6479	9.294 9839	8.745 4680	8.244 2370
15	10.739 5457	10.379 6580	10.037 5809	9.712 2490	9.107 9140	8.559 4787
16	11.234 0151	10.837 7696	10.462 1620	10.105 8953	9.446 6486	8.851 3692
17	11.707 1914	11.274 0663	10.864 6086	10.477 2597	9.763 2230	9.121 6381
18	12.159 9918	11.689 5869	11.246 0745	10.827 6035	10.059 0869	9.371 8871
19	12.593 2936	12.085 3209	11.607 6535	11.158 1165	10.335 5953	9.603 5992
20	13.007 9365	12.462 2103	11.950 3825	11.469 9212	10.594 0143	9.818 1474
21	13.404 7239	12.821 1527	12.275 2441	11.764 0766	10.835 5273	10.016 8032
22	13.784 4248	13.163 0026	12.583 1697	12.041 5817	11.061 2405	10.200 7437
23	14.147 7749	13.488 5739	12.875 0424	12.303 3790	11.272 1874	10.371 0590
24	14.495 4784	13.798 6418	13.151 6990	12.550 3575	11.469 3340	10.528 7583
25	14.828 2090	14.093 9446	13.413 9327	12.783 3562	11.653 5832	10.674 7762
26	15.146 6115	14.375 1853	13.662 4954	13.003 1662	11.825 7787	10.809 9780
27	15.451 3028	14.643 0336	13.898 0999	13.210 5341	11.986 7091	10.935 1648
28	15.742 8735	14.898 1273	14.121 4217	13.406 1643	12.137 1113	11.051 0785
29	16.021 8885	15.141 0736	14.333 1012	13.590 7210	12.277 6741	11.158 4060
30	16.288 8885	15.372 4510	14.533 7452	13.764 8312	12.409 0412	11.257 7833
31	16.544 3910	15.592 8105	14.723 9291	13.929 0860	12.531 8142	11.349 7994
32	16.788 8909	15.802 6767	14.904 1982	14.084 0434	12.646 5553	11.434 9994
33	17.022 8621	16.002 5492	15.075 0694	14.230 2296	12.753 7900	11.513 8884
34	17.246 7580	16.192 9040	15.237 0326	14.368 1411	12.854 0094	11.586 9337
35	17.461 0124	16.374 1943	15.390 5522	14.498 2464	12.947 6723	11.654 5682
36	17.666 0406	16.546 8517	15.536 0684	14.620 9871	13.035 2078	11.717 1928
37	17.862 2398	16.711 2873	15.673 9985	14.736 7803	13.117 0166	11.775 1785
38	18.049 9902	16.867 8927	15.804 7379	14.846 0192	13.193 4735	11.828 8690
39	18.229 6557	17.017 0407	15 928 6615	14.949 0747	13.264 9285	11.878 5824
40	18.401 5844	17.159 0864	16.046 1247	15.046 2969	13.331 7089	11.924 6133
41	18.566 1005	17.294 3680	16.157 4642	15.138 0159	13.394 1204	11.967 2346
42	18.723 5498	17.423 2076	16.262 9992	15.224 5433	13.452 4490	12.006 6987
43	18.874 2103	17.545 9120	16.363 0324	15.306 1729	13.506 9617	12.043 2395
44	19.018 3831	17.662 7733	16.457 8506	15.383 1820	13.557 9081	12.077 0736
45	19.156 3474	17.774 0698	16.547 7257	15.455 8321	13.605 5216	12.108 4015
46	19.288 3707	17.880 0665	16.632 9154	15.524 3699	13.650 0202	12.137 4088
47	19.414 7088	17.981 0157	16.713 6639	15.589 0282	13.691 6077	12.164 2674
48	19.535 6065	18.077 1578	16.790 2027	15.650 0266	13.730 4744	12.189 1365
49	19.651 2981	18.168 7217	16.862 7514	15.707 5723	13.766 7986	12.212 1634
50	19.762 0078	18.255 9255	16.931 5179	15.761 8606	13.800 7463	12.233 4846

$$\frac{1}{1-\frac{1}{(1+i)^n}}{i}$$

n	½%	1%	1¼%	1½%	1¾%	2%
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1.017 5000	1.020 0000
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	0.513 1630	0.515 0495
3	0.336 6722	0.340 0221	0.341 7012	0.343 3830	0.345 0675	0.346 7547
4	0.253 1328	0.256 2811	0.257 8610	0.259 4448	0.261 0324	0.262 6238
5	0.203 0100	0.206 0398	0.207 5621	0.209 0893	0.210 6214	0.212 1584
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	0.177 0226	0.178 5258
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	0.153 0306	0.154 5120
8	0.127 8289	0.130 6903	0.132 1331	0.133 5840	0.135 0429	0.136 5098
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	0.121 0581	0.122.5154
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	0.109 8754	0.111 3265
11	0.093 6590	0.096 4541	0.097 8684	0.099 2938	0.100 7304	0.102 1779
12	0.086 0664	0.088 8488	0.090 2583	0.091 6800	0.093 1138	0.094 5596
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	0.086 6728	0.088 1184
14	0.074 1361	0.076 9012	0.078 3052	0.079 7233	0.081 1556	0.082 6020
15	0.069 3644	0.072 1238	0.073 5265	0.074 9444	0.076 3774	0.077 8255
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	0.072 1996	0.073 6501
17	0.061 5058	0.064 2581	0.065 6602	0.067 0797	0.068 5162	0.069 9698
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	0.065 2449	0.066 7021
19	0.055 3025	0.058 0518	0.059 4555	0.060 8785	0.062 3206	0.063 7818
20	0.052 6664	0.055 4153	0.056 8204	0.058 2457	0.059 6912	0.061 1567
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	0.057 3146	0.058 7848
22	0.048 1138	0.050 8637	0.052 2724	0.053 7033	0.055 1564	0.056 6314
23	0.046 1346	0.048 8858	0.050 2967	0.051 7308	0.053 1880	0.054 6681
24	0.044 3206	0.047 0735	0.048 4866	0.049 9241	0.051 3857	0.052 8711
25	0.042 6519	0.045 4068	0.046 8225	0.048 2635	0.049 7295	0.051 2204
26	0.041 1116	0.043 8689	0.045 2873	0.046 7320	0.048 2027	0.049 6992
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	0.046 7908	0.048 2931
28	0.038 3617	0.041 1244	0.042 5486	0.044 0011	0.045 4815	0.046 9897
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	0.044 2642	0.045 7784
30	0.035 9789	0.038 7481	0.040 1785	0.041 6392	0.043 1298	0.044 6499
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	0.042 0701	0.043 5963
32	0.033 8945	0.036 6709	0.038 1079	0.039 5771	0.041 0781	0.042 6106
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	0.040 1478	0.041 6865
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	0.039 2736	0.040 8187
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	0.038 4508	0.040 0022
36	0.030 4219	0.033 2143	0.034 6653	0.036 1524	0.037 6751	0.039 2329
37	0.029 6714	0.032 4680	0.033 9227	0.035 4144	0.036 9426	0.038 5068
38	0.028 9604	0.031 7615	0.033 2198	0.034 7161	0.036 2499	0.037 8206
39	0.028 2861	0.031 0916	0.032 5536	0.034 0546	0.035 5940	0.037 1711
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	0.034 9721	0.036 5558
41	0.027 0363	0.029 8510	0.031 3206	0.032 8311	0.034 3817	0.035 9719
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	0.033 8206	0.035 4173
43	0.025 9032	0.028 7274	0.030 2047	0.031 7247	0.033 2867	0.034 8899
44	0.025 3754	0.028 2044	0.029 6856	0.031 2104	0.032 7781	0.034 3879
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	0.032 2932	0.033 9096
46	0.024 3889	0.027 2278	0.028 7168	0.030 2512	0.031 8304	0.033 4534
47	0.023 9273	0.026 7711	0.028 2641	0.029 8034	0.031 3884	0.033 0179
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	0.030 9657	0.032 6018
49	0.023 0609	0.025 9147	0.027 4156	0.028 9648	0.030 5612	0.032 2040
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	0.030 1739	0.031 8232

Note: For Rent of Annuity $\frac{1}{\frac{(1+i)^n-1}{i}}$ subtract the rate per cent.

Example: Rent of 1 at 1½% for 25 years is .0482635 — .015 = .0332635.

$$\frac{1}{1-\frac{1}{(1+i)^n}}{i}$$

n	2¼%	2½%	2¾%	3%	3½%	4%
1	1.022 5000	1.025 0000	1.027 5000	1.030 0000	1.035 0000	1.040 0000
2	0.516 9376	0.518 8272	0.520 7183	0.522 6108	0.526 4005	0.530 1961
3	0.348 4446	0.350 1372	0.351 8324	0.035 5304	0.356 9342	0.360 3485
4	0.264 2189	0.265 8179	0.267 4206	0.269 0271	0.272 2511	0.275 4901
5	0.213 7002	0.215 2469	0.216 7983	0.218 3546	0.221 4814	0.224 6271
6	0.180 0350	0.181 5500	0.183 0708	0.184 5975	0.187 6682	0.190 7619
7	0.156 0003	0.157 4954	0.158 9975	0.160 5064	0.163 5445	0.166 6096
8	0.137 9846	0.139 4674	0.140 9580	0.142 4564	0.145 4767	0.148 5278
9	0.123 9817	0.125 4569	0.126 9410	0.128 4339	0.131 4460	0.134 4930
10	0.112 7877	0.114 2588	0.115 7397	0.117 2305	0.120 2414	0.123 2909
11	0.103 6365	0.105 1060	0.106 5863	0.108 0775	0.111 0920	0.114 1490
12	0.096 0174	0.097 4871	0.098 9687	0.100 4621	0.103 4840	0.106 5522
13	0.089 5769	0.091 0483	0.092 5325	0.094 0295	0.097 0616	0.100 1437
14	0.084 0623	0.085 5365	0.087 0246	0.088 5263	0.091 5707	0.094 6690
15	0.079 2885	0.080 7665	0.082 2592	0.083 7666	0.086 8251	0.089 9411
16	0.075 1166	0.076 5990	0.078 0971	0.079 6109	0.082 6848	0.085 8200
17	0.071 4404	0.072 9278	0.074 4319	0.075 9525	0.079 0431	0.082 1985
18	0.068 1772	0.069 6701	0.071 1806	0.072 7087	0.075 8168	0.078 9933
19	0.065 2618	0.066 7606	0.068 2780	0.069 8139	0.072 9403	0.076 1386
20	0.062 6421	0.064 1471	0.065 6717	0.067 2157	0.070 3611	0.073 5818
21	0.060 2757	0.061 7873	0.063 3194	0.064 8718	0.068 0366	0.071 2801
22	0.058 1282	0.059 6466	0.061 1864	0.062 7474	0.065 9321	0.069 1988
23	0.056 1710	0.057 6964	0.059 2441	0.060 8139	0.064 0188	0.067 3091
24	0.054 3802	0.055 9128	0.057 4686	0.059 0474	0.062 2728	0.065 5868
25	0.052 7360	0.054 2759	0.055 8400	0.057 4279	0.060 6740	0.064 0120
26	0.051 2213	0.052 7688	0.054 3412	0.055 9383	0.059 2054	0.062 5674
27	0.049 8219	0.051 3769	0.052 9578	0.054 5642	0.057 8524	0.061 2385
28	0.048 5253	0.050 0879	0.051 6774	0.053 2932	0.056 6027	0.060 0130
29	0.047 3208	0.048 8913	0.050 4894	0.052 1147	0.055 4454	0.058 8799
30	0.046 1993	0.047 7776	0.049 3844	0.051 0193	0.054 3713	0.057 8301
31	0.045 1528	0.046 7390	0.048 3545	0.049 9989	0.053 3724	0.056 8554
32	0.044 1742	0.045 7683	0.047 3926	0.049 0466	0.052 4415	0.055 9486
33	0.043 2572	0.044 8594	0.046 4925	0.048 1561	0.051 5724	0.055 1036
34	0.042 3966	0.044 0068	0.045 6488	0.047 3220	0.050 7597	0.054 3148
35	0.041 5873	0.043 2056	0.044 8565	0.046 5393	0.049 9984	0.053 5773
36	0.040 8252	0.042 4516	0.044 1113	0.045 8038	0.049 2842	0.052 8869
37	0.040 1064	0.041 7409	0.043 4095	0.045 1116	0.048 6133	0.052 2396
38	0.039 4275	0.041 0701	0.042 7476	0.044 4593	0.047 9821	0.051 6319
39	0.038 7854	0.040 4362	0.042 1226	0.043 8439	0.047 3878	0.051 0608
40	0.038 1774	0.039 8362	0.041 5315	0.043 2624	0.046 8273	0.050 5235
41	0.037 6009	0.039 2679	0.040 9720	0.042 7124	0.046 2982	0.050 0174
42	0.037 0536	0.038 7288	0.040 4418	0.042 1917	0.045 7983	0.049 5402
43	0.036 5336	0.038 2169	0.039 9387	0.041 6981	0.045 3254	0.049 0899
44	0.036 0390	0.037 7304	0.039 4610	0.041 2299	0.044 8777	0.048 6645
45	0.035 5681	0.037 2675	0.039 0069	0.040 7852	0.044 4534	0.048 2625
46	0.035 1192	0.036 8268	0.038 5749	0.040 3625	0.044 0511	0.047 8821
47	0.034 6911	0.036 4067	0.038 1636	0.039 9605	0.043 6692	0.047 5219
48	0.034 2823	0.036 0060	0.037 7716	0.039 5778	0.043 3065	0.047 1807
49	0.033 8918	0.035 6235	0.037 3977	0.039 2131	0.042 9617	0.046 8571
50	0.033 5184	0.035 2581	0.037 0409	0.038 8655	0.042 6337	0.046 5502

Note: For Rent of Annuity $\frac{1}{\frac{(1+i)^n-1}{i}}$ subtract the rate per cent.

Example: Rent of 1 at 2% for 20 years is .0611567 — .02 = .0411567.

$$\frac{1}{1-\frac{1}{(1+i)^n}}{i}$$

n	4½%	5%	5½%	6%	7%	8%
1	1.045 0000	1.050 0000	1.055 0000	1.060 0000	1.070 0000	1.080 0000
2	0.533 9976	0.537 8049	0.541 6180	0.545 4369	0.553 0918	0.560 7692
3	0.363 7734	0.367 2086	0.370 6541	0.374 1098	0.381 0517	0.388 0335
4	0.278 7437	0.282 0118	0.285 2945	0.288 5915	0.295 2281	0.301 9208
5	0.227 7916	0.230 9748	0.234 1764	0.237 3964	0.243 8907	0.250 4564
6	0.193 8784	0.197 0175	0.200 1790	0.203 3626	0.209 7958	0.216 3154
7	0.169 7015	0.172 8198	0.175 9644	0.179 1350	0.185 5532	0.192 0724
8	0.151 6097	0.154 7218	0.157 8640	0.161 0359	0.167 4678	0.174 0148
9	0.137 5745	0.140 6901	0.143 8395	0.147 0222	0.153 4865	0.160 0797
10	0.126 3788	0.129 5046	0.132 6678	0.135 8680	0.142 3775	0.149 0295
11	0.117 2482	0.120 3889	0.123 5707	0.126 7929	0.133 3569	0.140 0763
12	0.109 6662	0.112 8254	0.116 0292	0.119 2770	0.125 9020	0.132 6950
13	0.103 2754	0.106 4558	0.109 6843	0.112 9601	0.119 6509	0.126 5218
14	0.097 8203	0.101 0240	0.104 2791	0.107 5849	0.114 3449	0.121 2968
15	0.093 1138	0.096 3423	0.099 6256	0.102 9628	0.109 7946	0.116 8295
16	0.089 0154	0.092 2699	0.095 5825	0.098 9521	0.105 8577	0.112 9769
17	0.085 4176	0.088 6999	0.092 0420	0.095 4448	0.102 4252	0.109 6294
18	0.082 2369	0.085 5462	0.088 9199	0.092 3565	0.099 4126	0.106 7021
19	0.079 4073	0.082 7450	0.086 1501	0.089 6209	0.096 7530	0.104 1276
20	0.076 8761	0.080 2426	0.083 6793	0.087 1846	0.094 3929	0.101 8522
21	0.074 6006	0.077 9961	0.081 4648	0.085 0046	0.092 2890	0.099 8322
22	0.072 5457	0.075 9705	0.079 4712	0.083 0456	0.090 4058	0.098 0321
23	0.070 6825	0.074 1368	0.077 6696	0.081 2785	0.088 7139	0.096 4222
24	0.068 9870	0.072 4709	0.076 0358	0.079 6790	0.087 1890	0.094 9780
25	0.067 4390	0.070 9525	0.074 5494	0.078 2267	0.085 8105	0.093 6788
26	0.066 0214	0.069 5643	0.073 1931	0.076 9044	0.084 5610	0.092 5071
27	0.064 7195	0.068 2919	0.071 9523	0.075 6972	0.083 4257	0.091 4481
28	0.063 5208	0.067 1225	0.070 8144	0.074 5926	0.082 3919	0.090 4889
29	0.062 4146	0.066 0455	0.069 7686	0.073 5796	0.081 4487	0.089 6185
30	0.061 3915	0.065 0514	0.068 8054	0.072 6489	0.080 5864	0.088 8274
31	0.060 4435	0.064 1321	0.067 9167	0.071 7922	0.079 7969	0.088 1073
32	0.059 5632	0.063 2804	0.067 0952	0.071 0023	0.079 0729	0.087 4508
33	0.058 7445	0.062 4900	0.066 3347	0.070 2729	0.078 4081	0.086 8516
34	0.057 9819	0.061 7554	0.065 6296	0.069 5984	0.077 7967	0.086 3041
35	0.057 2705	0.061 0717	0.064 9749	0.068 9739	0.077 2340	0.085 8033
36	0.056 6058	0.060 4345	0.064 3664	0.068 3948	0.076 7153	0.085 3447
37	0.055 9840	0.059 8398	0.063 7999	0.067 8574	0.076 2369	0.084 9244
38	0.055 4017	0.059 2842	0.063 2722	0.067 3581	0.075 7951	0.084 5389
39	0.054 8557	0.058 7646	0.062 7799	0.066 8938	0.075 3868	0.084 1851
40	0.044 3432	0.058 2782	0.062 3203	0.066 4615	0.075 0091	0.083 8602
41	0.053 8616	0.057 8223	0.061 8909	0.066 0589	0.074 6596	0.083 5615
42	0.053 4087	0.057 3947	0.061 4893	0.065 6834	0.074 3359	0.083 2868
43	0.052 9824	0.056 9933	0.061 1134	0.065 3331	0.074 0359	0.083 0341
44	0.052 5807	0.056 6163	0.060 7613	0.065 0061	0.073 7577	0.082 8015
45	0.052 2020	0.056 2617	0.060 4313	0.064 7005	0.073 4996	0.082 5873
46	0.051 8447	0.055 9282	0.060 1218	0.064 4149	0.073 2600	0.082 3899
47	0.051 5073	0.055 6142	0.059 8313	0.064 1477	0.073 0374	0.082 2080
48	0.051 1886	0.055 3184	0.059 5585	0.063 8977	0.072 8307	0.082 0403
49	0.050 8872	0.055 0397	0.059 3023	0.063 6636	0.072 6385	0.081 8856
50	0.050 6022	0.054 7767	0.059 0615	0.063 4443	0.072 4599	0.081 7429

Note: For Rent of Annuity $\frac{1}{\frac{(1+i)^n-1}{i}}$ subtract the rate per cent.

Example: Rent of 1 for 30 years at 3% is .0510193 — .03 = .0210193.

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